Convergence results for finite element and finite difference approximation of nonlocal fracture models

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Joint work with
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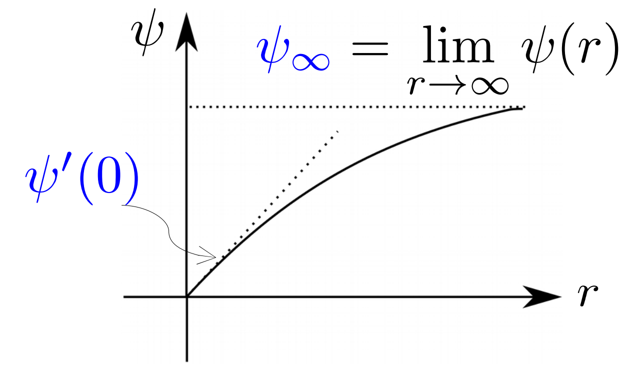
Overview of the talk

- Nonlinear peridynamic model
- Well-posedness of nonlocal model
- A priori convergence: Theory
- A priori convergence: Numerical results
- Recent and future works

Nonlinear peridynamic model

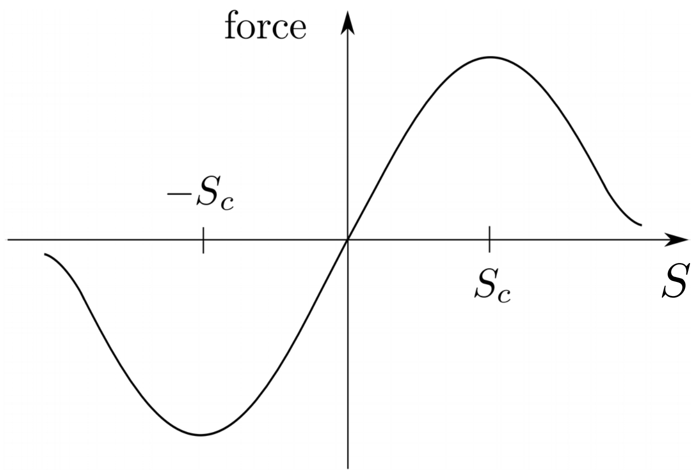
Let ϵ is the horizon, $B_\epsilon(\boldsymbol{x})$ ball of radius ϵ, and $\boldsymbol{u}(\boldsymbol{x})$ displacement of material point $\boldsymbol{x} \in D$. In this work, we consider linearized pairwise strain $S(\boldsymbol{y}, \boldsymbol{x}; \boldsymbol{u})$ given by

$$S(\boldsymbol{y}, \boldsymbol{x}; \boldsymbol{u}) = \frac{\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}(\boldsymbol{x})}{|\boldsymbol{y} - \boldsymbol{x}|} \cdot \frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|}.$$



Suppose $\hat{\boldsymbol{f}}^\epsilon(\boldsymbol{y}, \boldsymbol{x})$ denotes the force applied on $\boldsymbol{x}$ from the neighboring point $\boldsymbol{y}$. Then total force at $\boldsymbol{x}$ is given by

$$\boldsymbol{f}^\epsilon(\boldsymbol{x}) = \int_{B_\epsilon(\boldsymbol{x})} \hat{\boldsymbol{f}}^\epsilon(\boldsymbol{y}, \boldsymbol{x}) d\boldsymbol{y}$$



We consider pairwise force based on smooth and concave potential function ψ[1,2]

$$\hat{\boldsymbol{f}}^\epsilon(\boldsymbol{y}, \boldsymbol{x}) = \frac{1}{\epsilon |B_\epsilon(\boldsymbol{0})|} \frac{\partial_S \psi(|\boldsymbol{y} - \boldsymbol{x}| S(\boldsymbol{y}, \boldsymbol{x})^2)}{|\boldsymbol{y} - \boldsymbol{x}|} \frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|}$$

[1] R. Lipton (2014) Dynamic brittle fracture as a small horizon limit of peridynamics. Journal of Elasticity, 117(1) 21-50.

[2] R. Lipton (2016) Cohesive dynamics and brittle fracture. Journal of Elasticity, 124(2), pp.143-191.

Equation of motion

Equation of motion

$$\rho\ddot{\boldsymbol{u}}(\boldsymbol{x},t) = \boldsymbol{f}^\epsilon(\boldsymbol{x};\boldsymbol{u}(t)) + \boldsymbol{b}(\boldsymbol{x},t), \qquad \forall \boldsymbol{x} \in D, t \in [0,T]$$

Boundary condition

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{g}(\boldsymbol{x},t) \qquad \forall \boldsymbol{x} \in D_u, t \in [0,T]$$

$$\boldsymbol{b}(\boldsymbol{x},t) = \boldsymbol{f}_{ext}(\boldsymbol{x},t) \qquad \forall \boldsymbol{x} \in D_f, t \in [0,T]$$

$D_u, D_f \subset D$ are layer with finite volume (area in 2-d) on which displacement and external force, respectively, are specified. External force is applied in the form of body force.

Initial condition: $\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \dot{\boldsymbol{u}}(\boldsymbol{x},0) = \boldsymbol{v}_0(\boldsymbol{x})$ for all $\boldsymbol{x} \in D$.

Weak form: Multiplying peridynamic equation by smooth test function $\tilde{\boldsymbol{u}}$ such that $\tilde{\boldsymbol{u}} = \boldsymbol{0}$ on D_u, and integrating over D, and using nonlocal integration by parts, we get

$$(\rho\ddot{\boldsymbol{u}}(t), \tilde{\boldsymbol{u}}) + a^\epsilon(\boldsymbol{u}(t), \tilde{\boldsymbol{u}}) = (\boldsymbol{b}(t), \tilde{\boldsymbol{u}})$$

where

$$a^\epsilon(\boldsymbol{u},\boldsymbol{w}) = \frac{1}{\epsilon|B_\epsilon(\boldsymbol{0})|}\int_D \left[\int_{B_\epsilon(\boldsymbol{x})} \psi'(|\boldsymbol{y}-\boldsymbol{x}|S(\boldsymbol{u})^2)|\boldsymbol{y}-\boldsymbol{x}|S(\boldsymbol{u})S(\boldsymbol{w})d\boldsymbol{y}\right] d\boldsymbol{x}$$

Well-posedness of nonlinear peridynamic model

- Using the fact that nonlinear peridynamic force is bounded and Lipschitz continuous with respect to displacement field $\boldsymbol{u} \in L^2_0(D)$, the existence of solutions over any finite time domain $[0, T]$ is shown [1].

- To prove existence of solutions in more regular spaces, we introduce boundary function ω into Peridynamic force. $\omega(\boldsymbol{x}) = 1$ in the interior and smoothly decays to 0 as $\boldsymbol{x}$ approaches boundary ∂D.

- To perform apriori error analysis of finite difference approximation, we consider Hölder space $C_0^{0,\gamma}(D)$, $\gamma \in (0, 1]$. In [2] we show existence of solutions in Hölder space $C_0^{0,\gamma}(D)$. In [3] we extend the results to state-based peridynamic models.

- For apriori error analysis of finite element approximation using continuous piecewise linear elements, we consider natural space $H^2(D) \cap H_0^1(D)$. In [4] we show existence of solutions in $H^2(D) \cap H_0^1(D)$.

[1] R. Lipton (2016) Cohesive dynamics and brittle fracture. Journal of Elasticity, 124(2), pp.143-191.

[2] P.K. Jha and R. Lipton (2018) Numerical analysis of nonlocal fracture models in Holder space. SIAM Journal on Numerical Analysis, 56(2), pp.906-941.

[3] P.K. Jha and R. Lipton (2019) Numerical convergence of finite difference approximations for state based peridynamic fracture models. Computer Methods in Applied Mechanics and Engineering, 351(1), 184 – 225.

[4] P.K. Jha and R. Lipton (2018) Finite element approximation of nonlocal fracture models. arXiv preprint arXiv:1710.07661. **Under review** in Discrete and Continuous Dynamical Systems Series B.

Well-posedness of nonlinear peridynamic model

Let W be either $C_0^{0,\gamma}(D)$ or $H^2(D) \cap H_0^1(D)$ space. We assume $\boldsymbol{u} \in W$ is extended by zero outside D. Domain D is assumed to be sufficiently smooth (precise details in [1,2]). Two key steps to show existence:

- Obtain Lipschitz bound on peridynamic force in W.
- Using Lipshitz bound, show local existence of unique solutions. Show that local existence of unique solutions can be repeatedly applied to get global existence of solutions for any time domain $(-T, T)$.

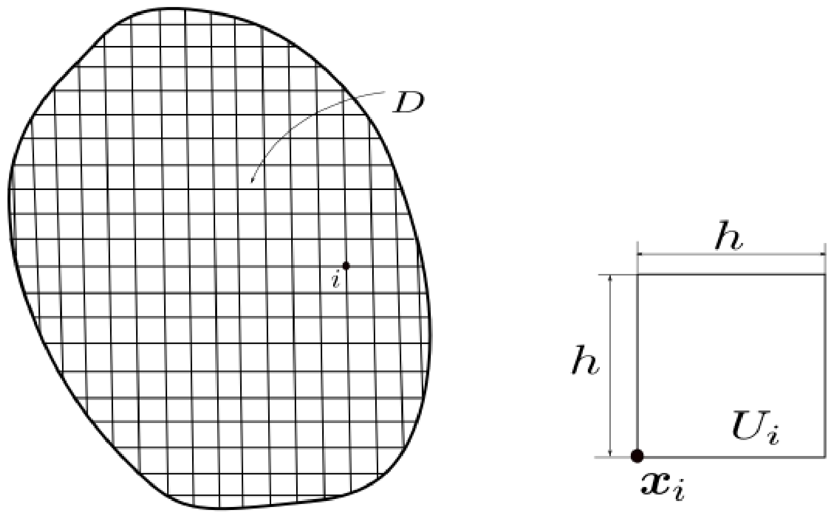
Theorem 1. ***Existence and uniqueness of solutions over finite time intervals***
Let $\boldsymbol{v}(t) = \dot{\boldsymbol{u}}(t)$, and $X = W \times W$. For any initial condition $x_0 \in X$, time interval $I_0 = (-T, T)$, and right hand side $\boldsymbol{b}(t)$ continuous in time for $t \in I_0$ such that $\boldsymbol{b}(t)$ satisfies $\sup_{t \in I_0} ||\boldsymbol{b}(t)||_W < \infty$, there is a unique solution $(\boldsymbol{u}(t), \boldsymbol{v}(t)) \in C^1(I_0; X)$ of Peridynamic equation of motion with initial condition x_0. Moreover, $(\boldsymbol{u}(t), \boldsymbol{v}(t))$ and $(\dot{\boldsymbol{u}}(t), \dot{\boldsymbol{v}}(t))$ are Lipschitz continuous in time for $t \in I_0$.

[1] P.K. Jha and R. Lipton (2018) Numerical analysis of nonlocal fracture models in Holder space. SIAM Journal on Numerical Analysis, 56(2), pp.906-941.
[2] P.K. Jha and R. Lipton (2019) Numerical convergence of finite difference approximations for state based peridynamic fracture models. Computer Methods in Applied Mechanics and Engineering, 351(1), 184 – 225.

Finite difference approximation

We approximate peridynamic equation using piecewise constant interpolation and central in time discretization. Let $\boldsymbol{u}_i^k$ denote the discrete displacement at mesh node $\boldsymbol{x}_i$ and time $t^k = k\Delta t$. We consider following piecewise constant function

$$\boldsymbol{u}_h^k(\boldsymbol{x}) = \sum_{i, \boldsymbol{x}_i \in D} \boldsymbol{u}_i^k \chi_{U_i}(\boldsymbol{x})$$



(a) (b)

Discrete problem is

$$\frac{\boldsymbol{u}_h^{k+1} - 2\boldsymbol{u}_h^k + \boldsymbol{u}_h^{k-1}}{\Delta t^2} = \boldsymbol{f}_h^{\epsilon}(t^k) + \boldsymbol{b}_h^k,$$

where

$$\boldsymbol{f}_h^{\epsilon}(\boldsymbol{x}, t^k) = \sum_{i, \boldsymbol{x}_i \in D} \boldsymbol{f}^{\epsilon}(\boldsymbol{x}_i, t^k) \chi_{U_i}(\boldsymbol{x}),$$

$$\boldsymbol{b}_h(\boldsymbol{x}, t^k) = \sum_{i, \boldsymbol{x}_i \in D} \boldsymbol{b}(\boldsymbol{x}_i, t^k) \chi_{U_i}(\boldsymbol{x})$$

Convergence of finite difference approximation

Error at time step k is defined as: $E^k = ||\boldsymbol{u}_h^k - \boldsymbol{u}(t^k)||$.

Theorem 2. *Let $\epsilon > 0$ be fixed. Let $(\boldsymbol{u}, \boldsymbol{v})$ be the solution of peridynamic equation. We assume $\boldsymbol{u}, \boldsymbol{v} \in C^2([0,T]; C^{0,\gamma}(D; \mathbb{R}^d))$. Then the finite difference scheme is consistent in both time and spatial discretization and converges to the exact solution uniformly in time with respect to the L^2 norm. If we assume the error at the initial step is zero then the error E^k at time t^k is bounded and satisfies*

$$\sup_{0 \leq k \leq T/\Delta t} E^k \leq O\left(C_t \Delta t + C_s \frac{h^\gamma}{\epsilon^2} \right),$$

where constant C_s and C_t are independent of h and Δt. Constants C_t, C_s depend on the ϵ and Hölder norm of the exact solution.

[1] P.K. Jha and R. Lipton (2018) Numerical analysis of nonlocal fracture models in Holder space. SIAM Journal on Numerical Analysis, 56(2), pp.906-941.
[2] P.K. Jha and R. Lipton (2019) Numerical convergence of finite difference approximations for state based peridynamic fracture models. Computer Methods in Applied Mechanics and Engineering, 351(1), 184 – 225.

Setting up peridynamic model

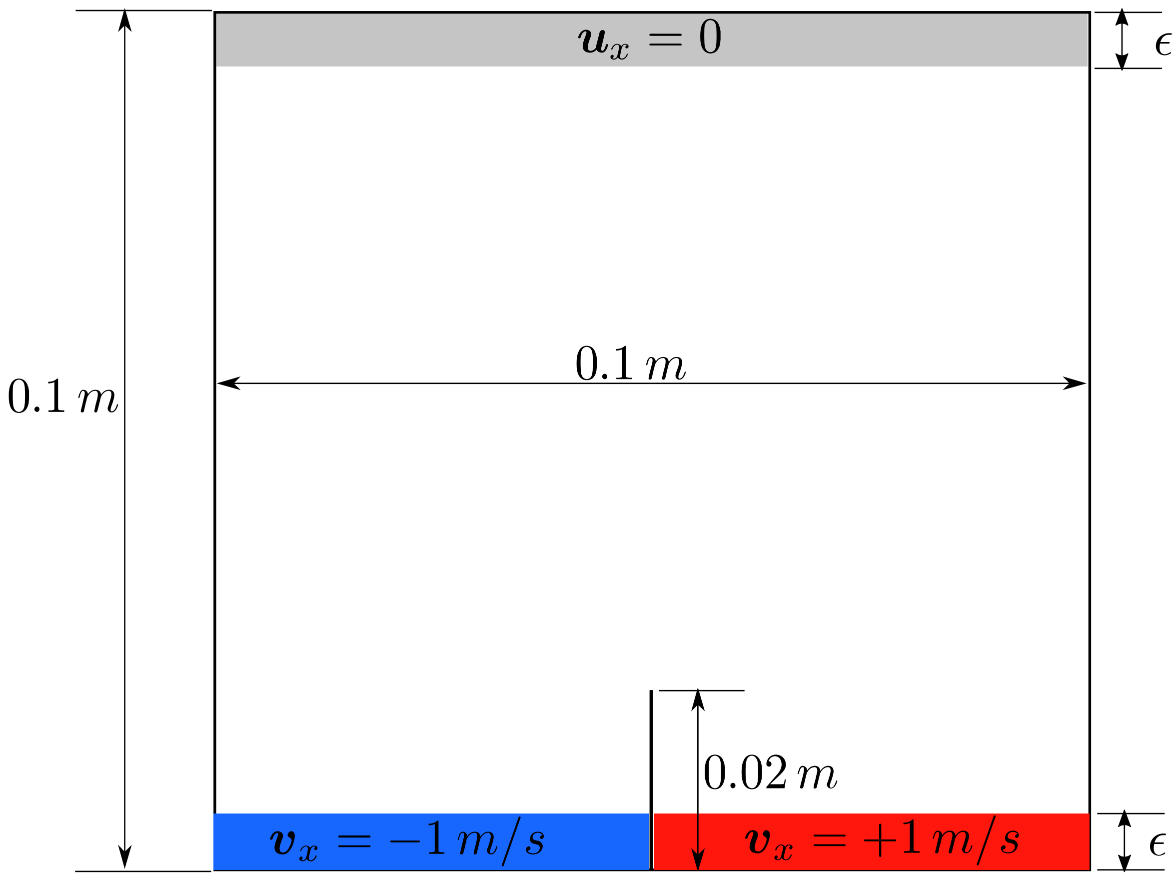
- Pairwise potential: $\psi(r) = c(1 - \exp[-\beta r^2])$
- Influence function: $J(r) = 1 - r$ for $0 \leq r < 1$ and $J(r) = 0$ for $r \geq 1$
- Critical strain: $S_c(\boldsymbol{y}, \boldsymbol{x}) = \dfrac{\pm \bar{r}}{\sqrt{|\boldsymbol{y} - \boldsymbol{x}|}}$, where $\bar{r}$ is the inflection point of function ψ
- We fix $\rho = 1200$ kg/m^3, bulk modulus $K = 25$ GPa, critical energy release rate $G_c = 500$ J/m^{-2}
- Using relation between nonlinear peridynamic model and linear elastic fracture mechanics[1], we find

$$c = 4712.4, \qquad \beta = 1.7533 \times 10^8, \qquad \bar{r} = \frac{1}{\sqrt{2\beta}} = 5.3402 \times 10^{-5}$$

[1] R. Lipton (2016) Cohesive dynamics and brittle fracture. Journal of Elasticity, 124(2), pp.143-191.

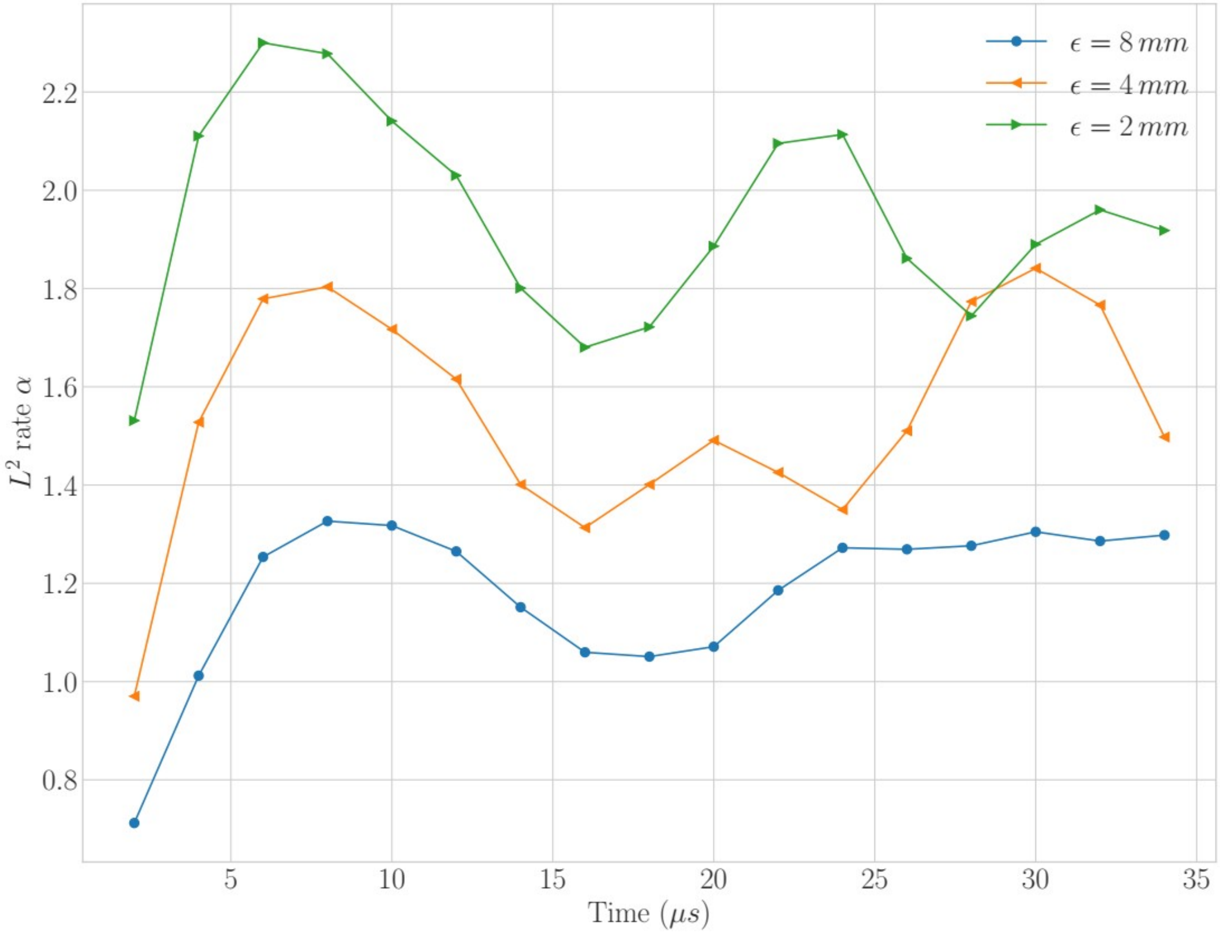
Mode I crack propagation: Setup

- Final time $T = 34\,\mu$s, time step $\Delta t = 0.004\,\mu$s
- Uniform grid on square domain $D = [0, 0.1\,\text{m}]^2$

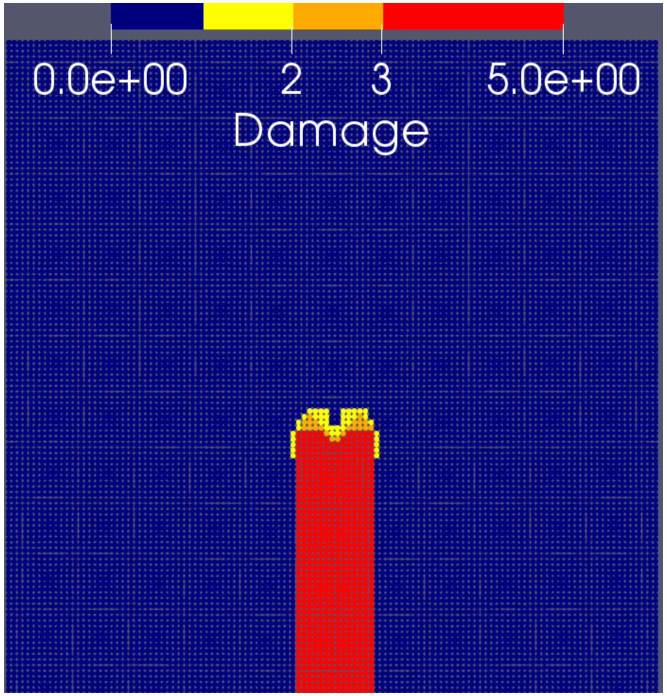


Convergence with respect to mesh size

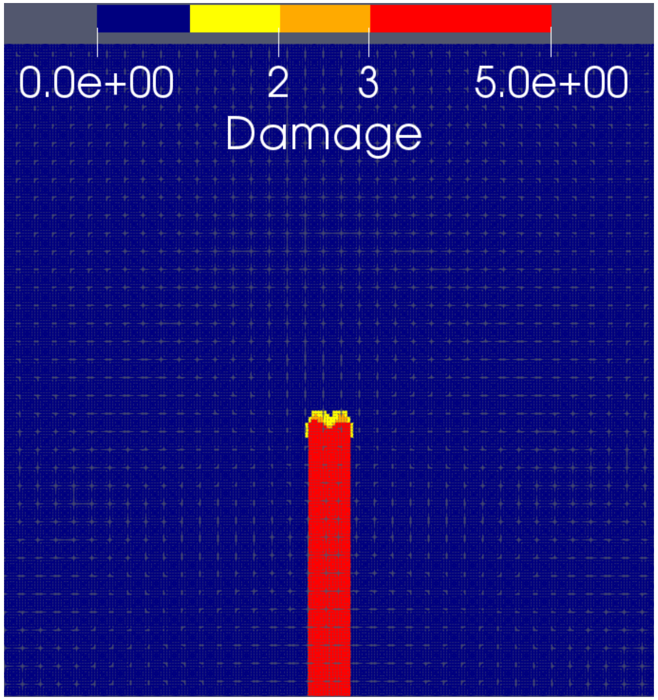
- Three set of horizons $\epsilon = 8, 4, 2$ mm. For each fixed ϵ, simulations were run with three different meshes of size $h = \epsilon/2, \epsilon/4, \epsilon/8$.



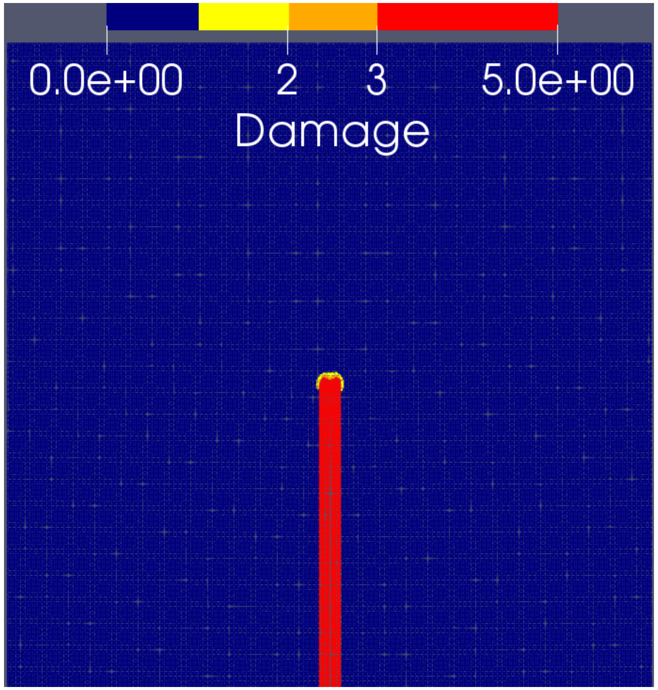
Localization of fracture zone



$\epsilon = 8$ mm



$\epsilon = 4$ mm



$\epsilon = 2$ mm

Finite element approximation

Let $V_h \subset H_0^1(D)$ be given by linear continuous interpolations over tetrahedral or triangular elements $\mathcal{T}_h$ where h denotes the size of finite element mesh. We assume elements are conforming and the mesh is shape regular.

For a continuous function $\boldsymbol{u}$ on $\bar{D}$, $\mathcal{I}_h(\boldsymbol{u})$ is the continuous piecewise linear interpolant on $\mathcal{T}_h$ and is given by

$$\mathcal{I}_h(\boldsymbol{u})(\boldsymbol{x}) = \sum_{T \in \mathcal{T}_h} \left[\sum_{i \in N_T} \boldsymbol{u}(\boldsymbol{x}_i) \phi_i(\boldsymbol{x}) \right].$$

Assuming that the size of each element in triangulation $\mathcal{T}_h$ is bounded by h, we have

$$||\boldsymbol{u} - \mathcal{I}_h(\boldsymbol{u})|| \le ch^2 ||\boldsymbol{u}||_2, \qquad \forall \boldsymbol{u} \in H_0^2(D; \mathbb{R}^d).$$

Projection: Let $\boldsymbol{r}_h(\boldsymbol{u}) \in V_h$ is the projection of $\boldsymbol{u} \in H^2(D) \cap H_0^1(D)$ such that

$$||\boldsymbol{u} - \boldsymbol{r}_h(\boldsymbol{u})|| = \inf_{\tilde{\boldsymbol{u}} \in V_h} ||\boldsymbol{u} - \tilde{\boldsymbol{u}}||$$

Central difference time discretization

$(\boldsymbol{u}_h^k, \boldsymbol{v}_h^k)$ and $(\boldsymbol{u}^k, \boldsymbol{v}^k)$ denote the approximate and the exact solution at k^{th} step. Projection is denoted as $(\boldsymbol{r}_h(\boldsymbol{u}^k), \boldsymbol{r}_h(\boldsymbol{v}^k))$. Approximate initial condition $\boldsymbol{u}_0, \boldsymbol{v}_0$ by their projection $\boldsymbol{r}_h(\boldsymbol{u}_0), \boldsymbol{r}_h(\boldsymbol{v}_0)$ and set $\boldsymbol{u}_h^0 = \boldsymbol{r}_h(\boldsymbol{u}_0)$, $\boldsymbol{v}_h^0 = \boldsymbol{r}_h(\boldsymbol{v}_0)$.

For $k \geq 1$, $(\boldsymbol{u}_h^k, \boldsymbol{v}_h^k)$ satisfies, for all $\tilde{\boldsymbol{u}} \in V_h$,

$$\left(\frac{\boldsymbol{u}_h^{k+1} - \boldsymbol{u}_h^k}{\Delta t}, \tilde{\boldsymbol{u}} \right) = (\boldsymbol{v}_h^{k+1}, \tilde{\boldsymbol{u}}),$$

$$\left(\frac{\boldsymbol{v}_h^{k+1} - \boldsymbol{v}_h^k}{\Delta t}, \tilde{\boldsymbol{u}} \right) = (\boldsymbol{f}^\epsilon(\boldsymbol{u}_h^k), \tilde{\boldsymbol{u}}) + (\boldsymbol{b}_h^k, \tilde{\boldsymbol{u}}),$$

where we denote projection of $\boldsymbol{b}(t^k)$, $\boldsymbol{r}_h(\boldsymbol{b}(t^k))$, as $\boldsymbol{b}_h^k$. Combining the two equations delivers central difference equation for $\boldsymbol{u}_h^k$. We have

$$\left(\frac{\boldsymbol{u}_h^{k+1} - 2\boldsymbol{u}_h^k + \boldsymbol{u}_h^{k-1}}{\Delta t^2}, \tilde{\boldsymbol{u}} \right) = (\boldsymbol{f}^\epsilon(\boldsymbol{u}_h^k), \tilde{\boldsymbol{u}}) + (\boldsymbol{b}_h^k, \tilde{\boldsymbol{u}}), \qquad \forall \tilde{\boldsymbol{u}} \in V_h.$$

Convergence of finite element approximation

Error at time step k is defined as: $E^k = ||\boldsymbol{u}_h^k - \boldsymbol{u}(t^k)||$.

Theorem 3. ***Convergence of Central difference approximation***
Let $(\boldsymbol{u}, \boldsymbol{v})$ be the exact solution of peridynamics equation and Let $(\boldsymbol{u}_h^k, \boldsymbol{v}_h^k)$ be the FE approximate solution. If $\boldsymbol{u}, \boldsymbol{v} \in C^2([0,T], H^2(D) \cap H_0^1(D))$, then the scheme is consistent and the error E^k satisfies following bound

$$\sup_{k \leq T/\Delta t} E^k = C_t \Delta t + C_s \frac{h^2}{\epsilon^2}$$

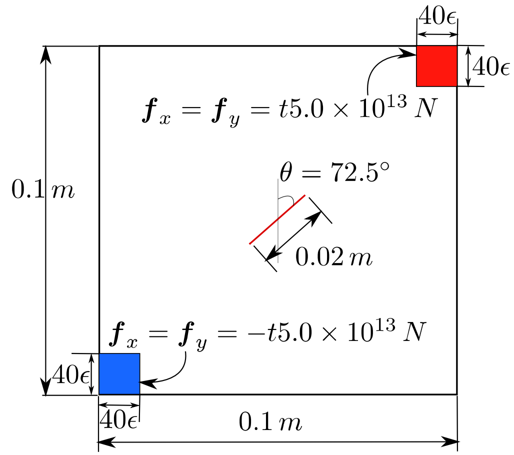
where constant C_t and C_s are independent of h and Δt and depends on the horizon and the norm of exact solution. Constant L/ϵ^2 is the Lipschitz constant of peridynamic force in L^2.

[1] P.K. Jha and R. Lipton (2018) Finite element approximation of nonlocal fracture models. arXiv preprint arXiv:1710.07661. **Under review** in Discrete and Continuous Dynamical Systems Series B.

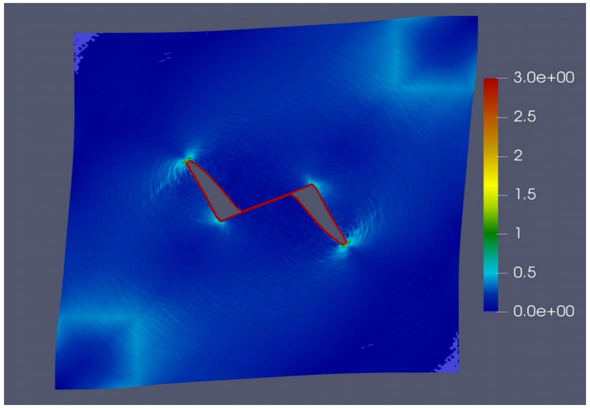
Recent work: Mix mode crack propagation

Material properties are same as in the Mode-I problem. We set

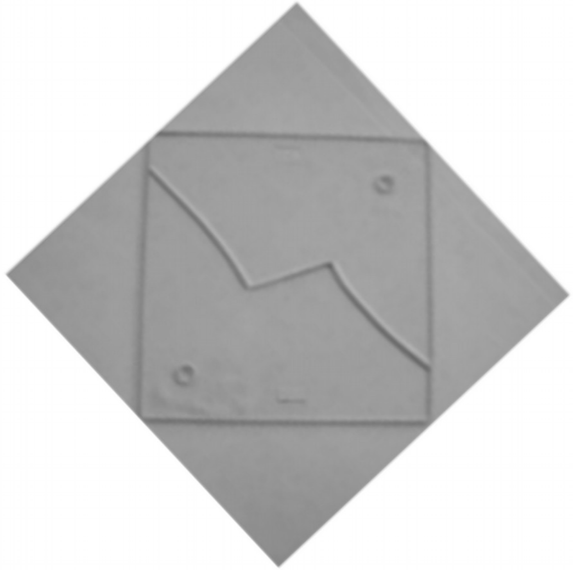
- Horizon $\epsilon = 0.5$ mm
- Mesh size $h = 0.125$ mm
- Final time $T = 140\ \mu$s
- Time step size $\Delta t = 0.004\ \mu$s



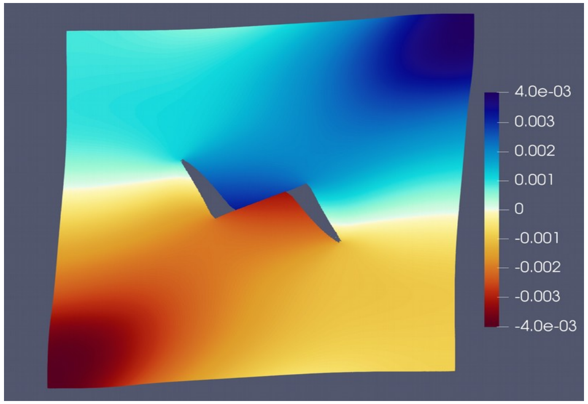
(a) Setup



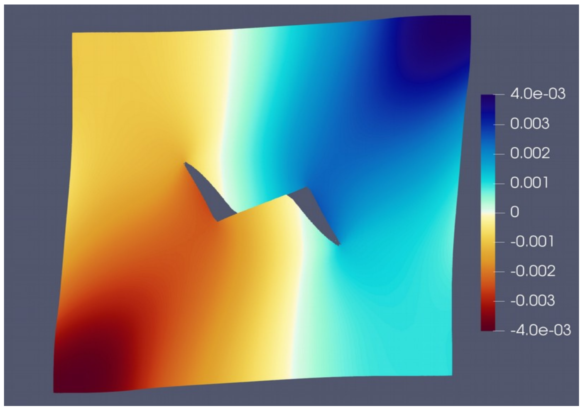
(b) Damage profile



(e) Experiment result [2]



(c) u_x plot



(d) u_y plot

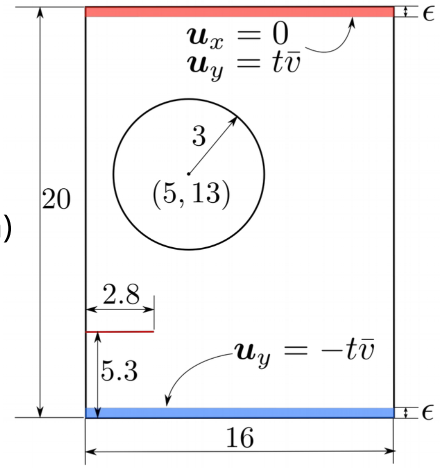
[1] R. Lipton, R. Lehoucq, & P.K. Jha (2019) Complex fracture nucleation and evolution with nonlocal elastodynamics. Journal of Peridynamics and Nonlocal Modeling. April 2019.

[2] M. R. Ayatollahi & M. R. M. Aliha (2009). Analysis of a new specimen for mixed mode fracture tests on brittle materials. Engineering Fracture Mechanics, 76(11), 1563-1573.

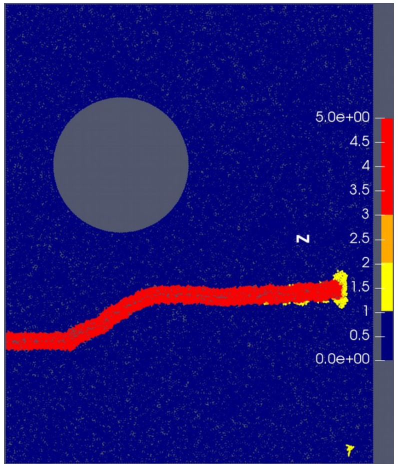
Recent work: Crack-void interaction

Material properties are same as in the Mode-I problem. We set

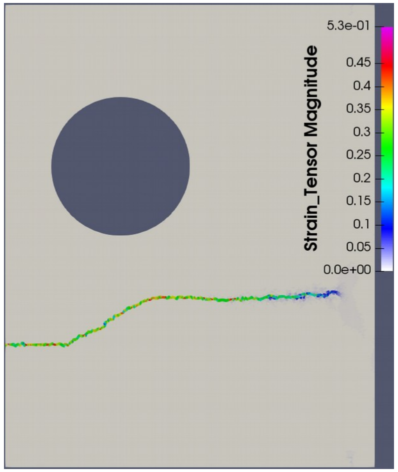
- Horizon $\epsilon = 0.4$ mm
- Mesh size $h = 0.1$ mm
- Final time $T = 800\,\mu$s
- Time step size $\Delta t = 0.004\,\mu$s



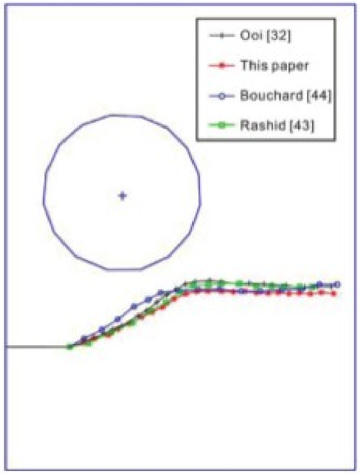
(a) Setup (units in mm)



(b) Damage profile



(c) Magnitude of symmetric gradient of displacement



(d) Numerical experiment results using FEM, Boundary element method [2]

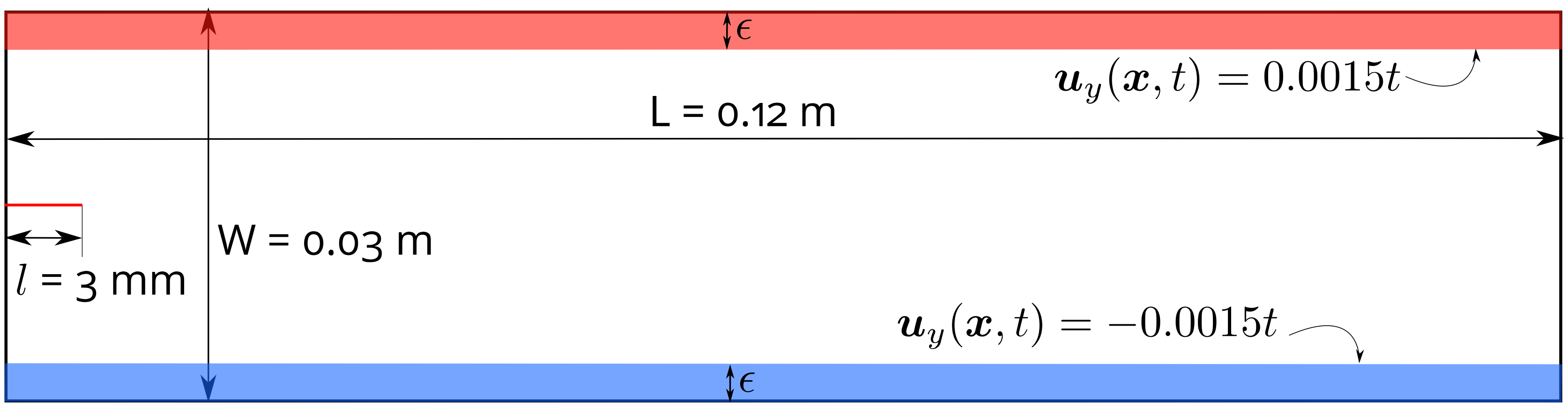
[1] P.K. Jha, P. Diehl & R. Lipton. Nodal finite element approximation of nonlocal fracture models. ***In preparation***.

[2] S. Dai,C. Augarde, C. Du & D. Chen (2015). A fully automatic polygon scaled boundary finite element method for modelling crack propagation. Engineering Fracture Mechanics, 133, 163-178.

Recent work: Wave reflection effect on crack velocity

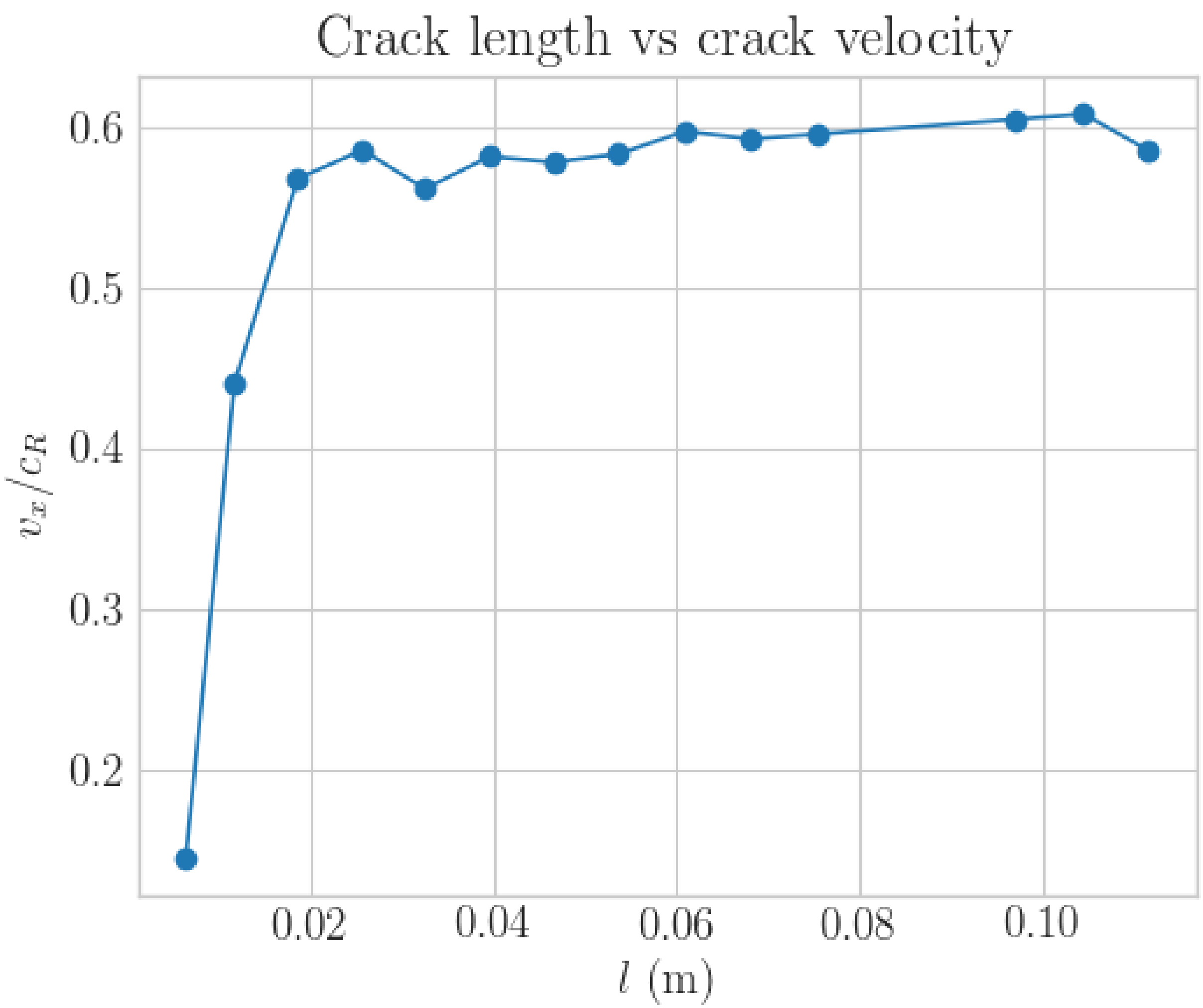
We consider a softer material with shear modulus $G = 35.2$ kPa, density $\rho = 1011$ kg/m^3, and critical energy release rate $G_c = 20$ J/m^{-2}. Poisson ratio is fixed to $\mu = 0.25$. Domain is $D = [0, 0.12\,\text{m}] \times [0, 0.03\,\text{m}]$.

- Horizon $\epsilon = 0.6$ mm, mesh size $h = 0.15$ mm
- Time $T = 1.1$ s, $\Delta t = 2.2\,\mu$s



Recent work: Wave reflection effect on crack velocity

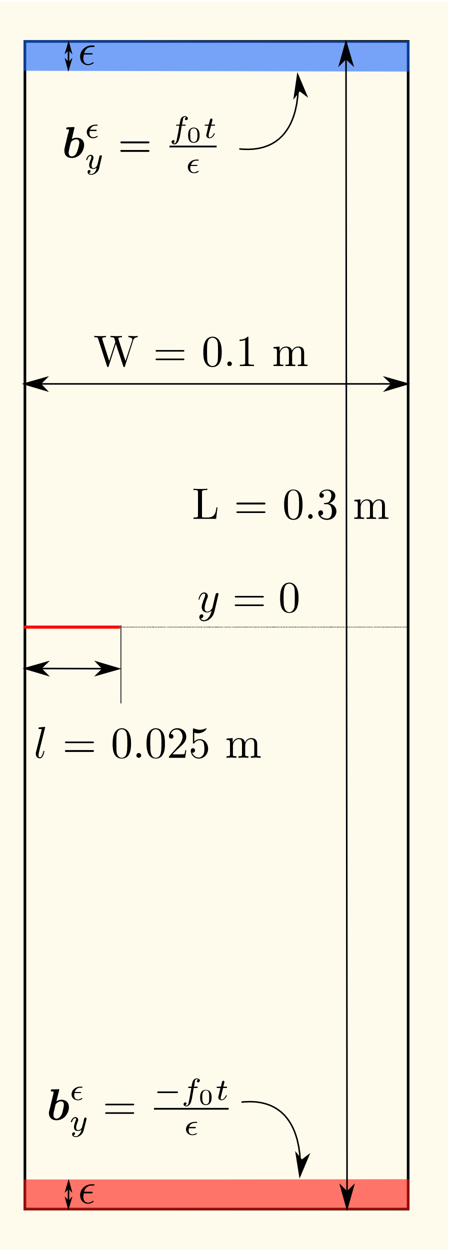
- Max crack length = 0.12 m
- Rayleigh wave speed $c_R = 5.502$ m/s



Recent work: J-integral

We consider a material with Young's modulus $E = 3.24$ GPa, density $\rho = 1200$ kg/m^3, and critical energy release rate $G_c = 500$ J/m^{-2}. We set

- Horizon $\epsilon = 1.25$ mm, mesh size $h = 0.3125$ mm
- Time $T = 800\,\mu$ s, $\Delta t = 0.016\,\mu$s
- $f_0 = 1.0 \times 10^{10}$



The energy associated to crack is given by

$$E(t) = \frac{1}{|B_\epsilon(\mathbf{0})|} \int_{A^c(t)} \int_{A(t)\cap B_\epsilon(\boldsymbol{x})} \partial_S W(S(\boldsymbol{y},\boldsymbol{x};\boldsymbol{u}(t))) \frac{\boldsymbol{y}-\boldsymbol{x}}{|\boldsymbol{y}-\boldsymbol{x}|} \cdot (\dot{\boldsymbol{u}}(\boldsymbol{x},t) + \dot{\boldsymbol{u}}(\boldsymbol{y},t)) d\boldsymbol{y} d\boldsymbol{x}.$$

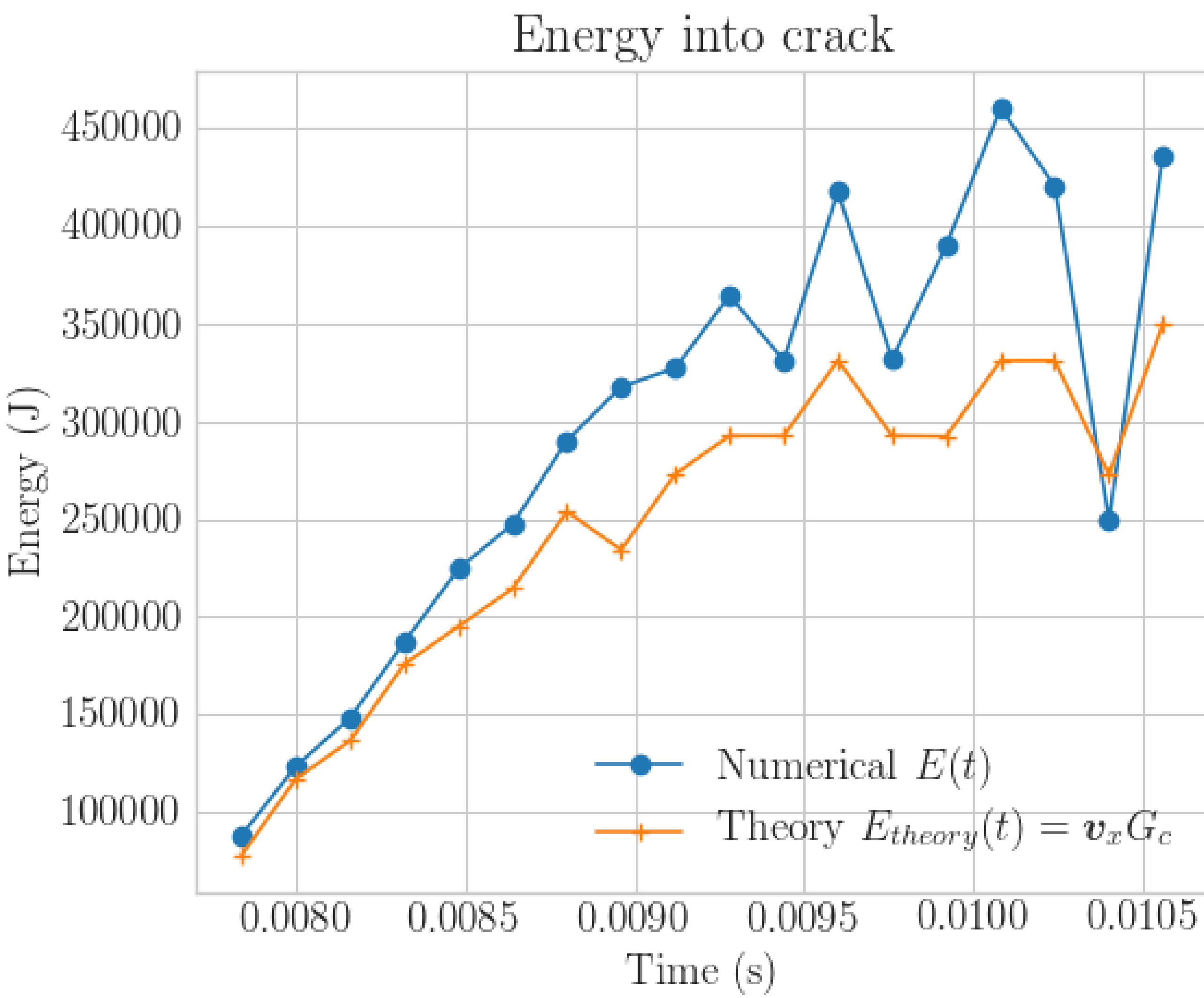
Here W is the peridynamic pairwise energy density. $A(t)$ is the rectangle domain with crack tip at its center. It is moving with tip. $A^c(t)$ is the complement of $A(t)$.

J-integral: In [1], we propose J-integral $J(t) := \frac{E(t)}{|\boldsymbol{v}|}$

[1] P.K. Jha & R. Lipton (2019). The relation of nonlocal cohesive models to classic dynamic fracture models: The single edge notch in tension. ***In preparation***.

Recent work: Energy into crack

We compute energy into crack using $E(t)$. Theoretically the energy into crack is given by $E_{theory} = |\boldsymbol{v}|G_c$, where $|\boldsymbol{v}|$ is the magnitude of crack velocity. In the current example, $|\boldsymbol{v}| = \boldsymbol{v}_x$.



Ongoing and future works

- In [1] we show that the classical kinetic relation is embedded in peridynamics and we have $\lim_{\epsilon \to 0} J(t) = G_c$, where $J(t)$ is the nonlocal J-integral (defined in slide 18). In LEFM, the classical kinetic relation for the crack velocity is postulated. In contrast, we obtain the classical kinetic relation from the Peridynamics.

- Open source computational library for nonlocal modeling. This is a joint work with Patrick Diehl (LSU) and Robert Lipton (LSU).

- Study of granular material using nonlinear nonlocal model.

[1] P.K. Jha & R. Lipton (2019). The relation of nonlocal cohesive models to classic dynamic fracture models: The single edge notch in tension. ***In preparation***.

Thank you!