Convergence results for finite element and finite difference approximation of nonlocal fracture models

Prashant K. Jha
prashant.j16o@gmail.com
jha@math.lsu.edu

Joint work with

Dr. Robert Lipton

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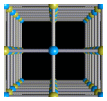


Outline of talk

- Peridynamic: Introduction
- Finite difference approximation and results
- Finite element approximation and results
- Summary and future works

Introduction

Let D be the material domain, D_c be nonlocal boundary, and $\mathbf{u}$ be the displacement field.

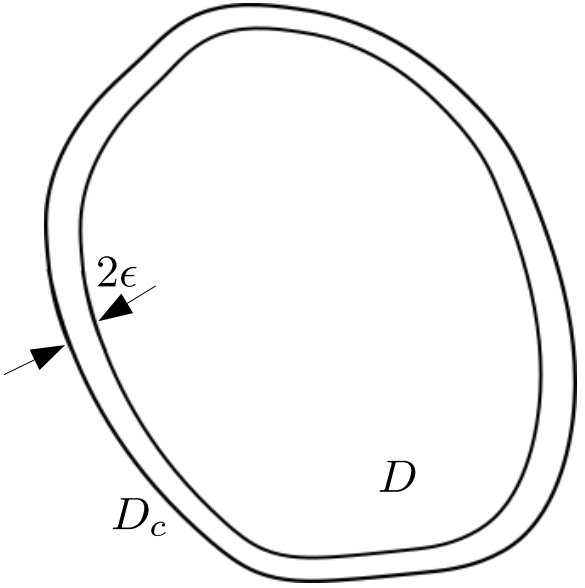
Let $\mathbf{x}$ denote the material point and $\chi(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x})$ denote the deformed position.

Strain between two material point $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\mathsf{S}(\mathbf{y}, \mathbf{x}; \mathbf{u}) = \frac{|\mathbf{y} + \mathbf{u}(\mathbf{y}) - \mathbf{x} - \mathbf{u}(\mathbf{x})| - |\mathbf{y} - \mathbf{x}|}{|\mathbf{y} - \mathbf{x}|}$$

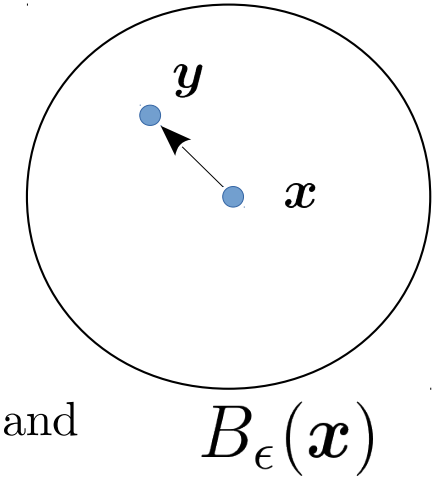
Assuming that displacement is small compared to the size of material, we linearize S and get

$$\mathsf{S}(\mathbf{y}, \mathbf{x}; \mathbf{u}) = \frac{\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})}{|\mathbf{y} - \mathbf{x}|} \cdot \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|}$$



Neighborhood of material point

Consider a material point $\mathbf{x}$. We introduce a length scale ϵ which is called size of horizon. This controls the extent of nonlocal interaction in the material.



Hydrostatic strain (average strain) at material point $\mathbf{x}$ is expressed as

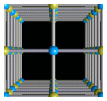
$$\theta(\mathbf{x};\mathbf{u}) = \frac{1}{|B_\epsilon(\mathbf{0})|}\int_{B_\epsilon(\mathbf{x})} J^\epsilon(|\mathbf{y}-\mathbf{x}|)S(\mathbf{y},\mathbf{x};\mathbf{u})|\mathbf{y}-\mathbf{x}|d\mathbf{y}$$

$J^\epsilon(|\mathbf{y}-\mathbf{x}|)$ is the influnece function. We assume $J^\epsilon(|\mathbf{y}-\mathbf{x}|) = J(|\mathbf{y}-\mathbf{x}|/\epsilon)$ and function J satisfies $0 \le J(r) \le M$ for $r < 1$ and $J(r) = 0$ for $r \ge 1$.

Generic form of force at $\mathbf{x}$ in peridynamic model is given by

$$\mathbf{F}^\epsilon(\mathbf{x};\mathbf{u}) = \frac{1}{|B_\epsilon(\mathbf{x})|}\int_{B_\epsilon(\mathbf{x})} \hat{\mathbf{f}}^\epsilon(\mathbf{y},\mathbf{x};\mathbf{u})d\mathbf{y}$$

$\hat{\mathbf{f}}^\epsilon$ depends on choice of ϵ and includes force due to pairwise interaction and force due to volume deformation.

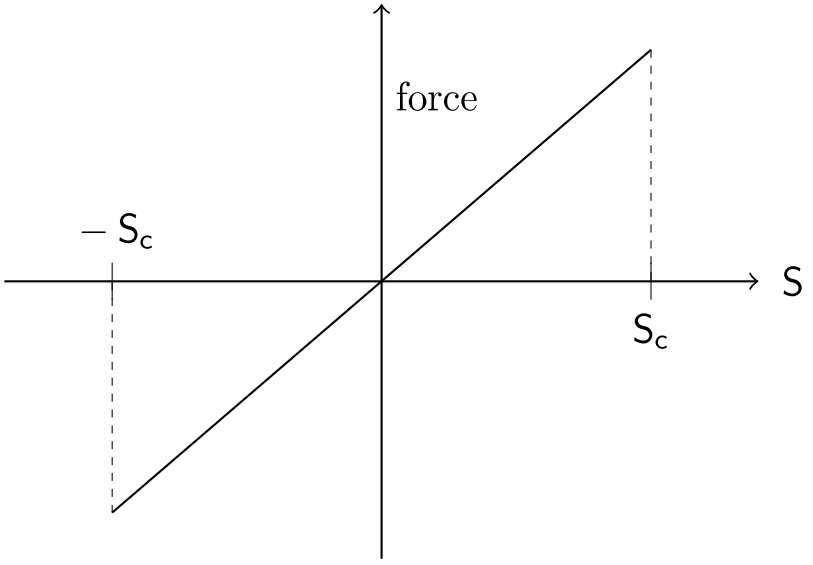


Example of pairwise force

- Example: Prototype Microelastic Brittle (PMB) material **Silling & Askari (2005), Bobaru & Hu (2012)**

$$\hat{\mathbf{f}}^{\epsilon}(\mathbf{y},\mathbf{x},\mathbf{u}) = \mu(S(\mathbf{y},\mathbf{x};\mathbf{u}))4\frac{J^{\epsilon}(|\mathbf{y}-\mathbf{x}|)}{\epsilon}S(\mathbf{y},\mathbf{x};\mathbf{u})\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}$$

where $\mu(\mathsf{S}) = 1$ if $|\mathsf{S}| < \mathsf{S_c}$ and $\mu(\mathsf{S}) = 0$ when $|\mathsf{S}| \geq \mathsf{S_c}$.



- If $\mathbf{u} \in \mathsf{C}^3(\mathsf{D};\mathbb{R}^{\mathsf{d}})$, and $\sup_{\mathbf{x}\in\mathsf{D}} |\boldsymbol{\nabla}^3\mathbf{u}(\mathbf{x})| < \infty$ then

$$\sup_{x\in D} |\mathbf{f}^{\epsilon}(\mathbf{x};\mathbf{u}) - \boldsymbol{\nabla}\cdot\bar{\mathbb{C}}\mathcal{E}\mathbf{u}(\mathbf{x})| = O(\epsilon^2), \quad \bar{\mathbb{C}} = \frac{2}{|B_1(\mathbf{0})|}\int_{B_1(\mathbf{0})} J(|\boldsymbol{\xi}|)\mathbf{e}_{\boldsymbol{\xi}}\otimes\mathbf{e}_{\boldsymbol{\xi}}\otimes\mathbf{e}_{\boldsymbol{\xi}}\otimes\mathbf{e}_{\boldsymbol{\xi}}|\boldsymbol{\xi}|d\boldsymbol{\xi},$$

$\mathbf{e}_{\boldsymbol{\xi}} = \boldsymbol{\xi}/|\boldsymbol{\xi}|$ and the strain tensor is $\mathcal{E}\mathbf{u}(\mathbf{x}) = (\boldsymbol{\nabla}\mathbf{u}(\mathbf{x}) + \boldsymbol{\nabla}\mathbf{u}^{\mathsf{T}}(\mathbf{x}))/2$.

Silling & Askari (2005): A meshfree method based on the peridynamic model of solid mechanics. Computers and Structures, vol 83, pp 1526-1535.
Bobaru & Hu (2012): The meaning, selection, and use of the peridynamic horizon and its relation to crack branching in brittle materials. Int. J. of Fracture, vol 176(2), pp 215-222.

Regularized pairwise force

- We consider peridynamic force of the form **Lipton (2014)**

$$\mathbf{F}^{\epsilon}(\mathbf{x};\mathbf{u}) = \frac{2}{|B_{\epsilon}(\mathbf{0})|}\int_{B_{\epsilon}(\mathbf{x})} \frac{J^{\epsilon}(|\mathbf{y}-\mathbf{x}|)}{\epsilon\sqrt{|\mathbf{y}-\mathbf{x}|}}\psi'(\sqrt{|\mathbf{y}-\mathbf{x}|}S)\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}d\mathbf{y}$$

Potential function ψ is multiwell function with one well at zero and other at $\pm\infty$

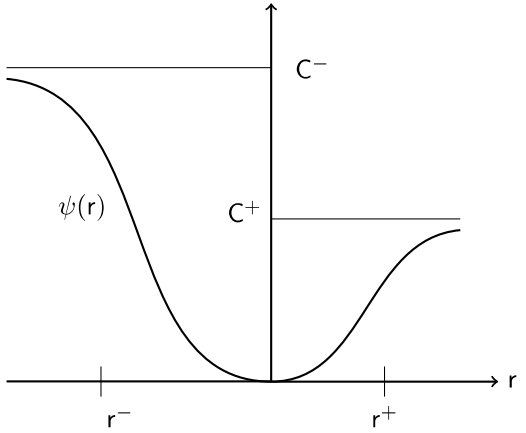


Fig: Profile of ψ

- Critical strain: $\mathsf{S}_{\mathsf{c}}^{+}(\mathbf{y},\mathbf{x}) = \frac{\mathsf{r}^{+}}{\sqrt{|\mathbf{y}-\mathbf{x}|}}, \quad \mathsf{S}_{\mathsf{c}}^{-}(\mathbf{y},\mathbf{x}) = \frac{\mathsf{r}^{-}}{\sqrt{|\mathbf{y}-\mathbf{x}|}}$

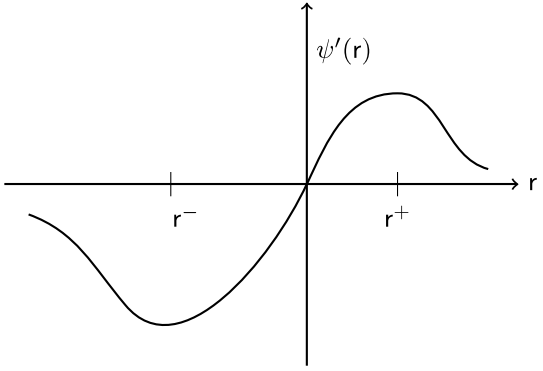


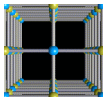
Fig: Profile of ψ'

- Example: $\psi(\mathsf{r}) = \mathsf{c}(1-\exp[-\beta\mathsf{r}^2])$

c, β are determined from elastic and fracture properties of material

$$\mathsf{r}^{+} = -\mathsf{r}^{-} = \frac{1}{\sqrt{2\beta}}$$

Lipton (2014): Dynamic brittle fracture as a small horizon limit of peridynamics. Journal of Elasticity, vol 117(1).



State-based peridynamic force

- We consider peridynamic force of the form **Lipton et al (2018)**

$$\mathbf{F}^{\epsilon}(\mathbf{x};\mathbf{u}) = \frac{1}{|B_{\epsilon}(\mathbf{0})|}\int_{B_{\epsilon}(\mathbf{x})} \frac{J^{\epsilon}(|\mathbf{y}-\mathbf{x}|)}{\epsilon^2}(g'(\theta(\mathbf{y};\mathbf{u})) + g'(\theta(\mathbf{x};\mathbf{u})))\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}d\mathbf{y}$$

Potential function g can either be multiwell function, similar to ψ, or it can be quadratic function giving linear force for hydrostatic strain θ

- Example: In this work we consider $\mathsf{g}(\mathsf{r}) = \bar{\mathsf{C}}\mathsf{r}^2/2$

$\bar{\mathsf{C}}$ is determined from elastic properties of material

Lipton, Said, & Jha (2018): Dynamic brittle fracture from nonlocal double-well potentials: A state-based model. Handbook of Nonlocal Continuum Mechanics for Materials and Structures, 1–27.

Peridynamic energy and equation of motion

- Pairwise energy density

  $$\mathcal{W}^\epsilon(S(\mathbf{y},\mathbf{x};\mathbf{u}(t))) = J^\epsilon(|\mathbf{y}-\mathbf{x}|)\frac{1}{\epsilon|\mathbf{y}-\mathbf{x}|}\psi(\sqrt{|\mathbf{y}-\mathbf{x}|}S(\mathbf{y},\mathbf{x};\mathbf{u}(t)))$$

- State-based energy density $\mathcal{V}^\epsilon(\theta(\mathbf{x};\mathbf{u}(t))) = \dfrac{g(\theta(\mathbf{x};\mathbf{u}(t)))}{\epsilon^2}$

- Total peridynamic energy

  $$\mathsf{PD}^\epsilon(\mathbf{u}) = \frac{1}{|\mathsf{B}_\epsilon(\mathbf{0})|}\int_{\mathsf{D}}\int_{\mathsf{B}_\epsilon(\mathbf{x})}|\mathbf{y}-\mathbf{x}|\mathcal{W}^\epsilon(\mathsf{S}(\mathbf{y},\mathbf{x};\mathbf{u}(\mathsf{t})))\,\mathsf{d}\mathbf{y}\mathsf{d}\mathbf{x} + \int_{\mathsf{D}}\mathcal{V}^\epsilon(\theta(\mathbf{x};\mathbf{u}(\mathsf{t})))\,\mathsf{d}\mathbf{x}$$

- Lagrangian $\mathrm{L}(\mathbf{u},\dot{\mathbf{u}},t) = \dfrac{\rho}{2}||\dot{\mathbf{u}}||^2_{L^2(D;\mathbb{R}^d)} - PD^\epsilon(\mathbf{u}) + \displaystyle\int_D \mathbf{b}\cdot\mathbf{u}d\mathbf{x}$

- Applying the principal of least action gives the nonlocal dynamics

  $$\rho\ddot{\mathbf{u}}(\mathbf{x},\mathsf{t}) = \mathbf{F}^\epsilon(\mathbf{x},\mathsf{t}) + \mathbf{b}(\mathbf{x},\mathsf{t}),\ \text{for } \mathbf{x}\in\mathsf{D}$$

- Initial condition $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x})$

- Boundary condition $\mathbf{u}(\mathbf{x},\mathsf{t}) = \mathbf{0}$ for $\mathbf{x}\in\mathsf{D}_\mathsf{c}$

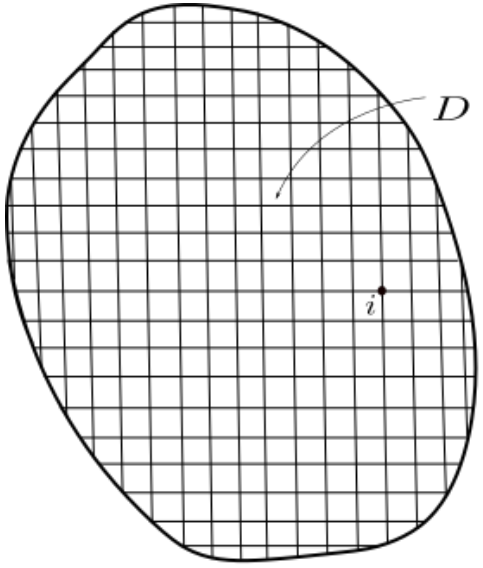
- For $\mathbf{b}\in\mathsf{C}^1([0,\mathsf{T}];\mathsf{L}^\infty(\mathsf{D};\mathbb{R}^\mathsf{d}))$ and initial data $\mathbf{u}_0,\mathbf{v}_0\in\mathsf{L}^\infty(\mathsf{D};\mathbb{R}^\mathsf{d})$, solution exist in $\mathsf{C}^2([0,\mathsf{T}];\mathsf{L}^\infty(\mathsf{D};\mathbb{R}^\mathsf{d}))$ **Lipton et al (2018)**

Lipton, Said, and Jha (2018): Dynamic brittle fracture from nonlocal double-well potentials: A state-based model. Handbook of Nonlocal Continuum Mechanics for Materials and Structures, 1–27.

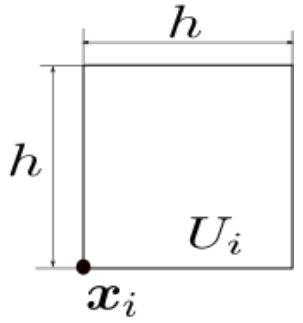
Finite difference approximation

Approximate peridynamic equation using piecewise constant interpolation and central in time discretization. Let $\mathbf{u}_i^k$ denote the discrete displacement at mesh node $\mathbf{x}_i$ and time $t^k = k\Delta t$. We consider following piecewise constant function

$$\mathbf{u}_h^k(\mathbf{x}) = \sum_{i,\mathbf{x}_i \in D} \mathbf{u}_i^k \chi_{U_i}(\mathbf{x})$$



(a)



(b)

Discrete problem is

$$\frac{\mathbf{u}_h^{k+1} - 2\mathbf{u}_h^k + \mathbf{u}_h^{k-1}}{\Delta t^2} = \mathbf{F}_h(t^k) + \mathbf{b}_h^k,$$

where

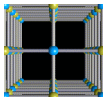
$$\mathbf{F}_h(\mathbf{x}, t^k) = \sum_{i,\mathbf{x}_i \in D} \mathbf{F}(\mathbf{x}_i, t^k)\chi_{U_i}(\mathbf{x}),$$

$$\mathbf{b}_h(\mathbf{x}, t^k) = \sum_{i,\mathbf{x}_i \in D} \mathbf{b}(\mathbf{x}_i, t^k)\chi_{U_i}(\mathbf{x})$$

Writing discrete problem in two first order difference equation

$$\frac{\mathbf{u}_h^{k+1} - \mathbf{u}_h^k}{\Delta t} = \mathbf{v}_h^{k+1}$$

$$\frac{\mathbf{v}_h^{k+1} - \mathbf{v}_h^k}{\Delta t} = \mathbf{F}_h(t^k) + \mathbf{b}_h^k$$



A priori convergence

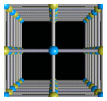
- Well-posedness of solutions in Hölder space $C^{0,\gamma}(D;\mathbb{R}^d)$ and existence of solutions in $C^2([0,T];C^{0,\gamma}(D;\mathbb{R}^d))$ is shown in **Jha & Lipton (2018a)**
- $E^k := ||\mathbf{u}_h^k - \mathbf{u}(t^k)|| + ||\mathbf{v}_h^k - \mathbf{v}(t^k)||$

Theorem 1. *Let $\epsilon > 0$ be fixed. Let $(\mathbf{u},\mathbf{v})$ be the solution of peridynamic equation. We assume $\mathbf{u},\mathbf{v} \in C^2([0,T];C^{0,\gamma}(D;\mathbb{R}^d))$. Then the finite difference scheme is consistent in both time and spatial discretization and converges to the exact solution uniformly in time with respect to the L^2 norm. If we assume the error at the initial step is zero then the error E^k at time t^k is bounded and satisfies*

$$\sup_{0\le k\le T/\Delta t} E^k \le O\left(C_t\Delta t + C_s\frac{h^\gamma}{\epsilon^2}\right),$$

where constant C_s and C_t are independent of h and Δt and C_s depends on the Hölder norm of the solution and C_t depends on the L^2 norms of time derivatives of the solution.

Jha & Lipton (2018a): Numerical convergence of finite difference approximations for state based peridynamic fracture models. Currently under review in CMAME. Arxiv: https://arxiv.org/abs/1805.00296



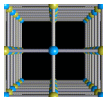
Numerical experiments: Material properties

- Pairwise potential: $\psi(\mathsf{r}) = \mathsf{c}(1 - \exp[-\beta \mathsf{r}^2])$
- State-based potential: $\mathsf{g}(\mathsf{r}) = \bar{\mathsf{C}}\mathsf{r}^2/2$
- Influence function: $\mathsf{J}(\mathsf{r}) = 1 - \mathsf{r}$ for $0 \leq \mathsf{r} < 1$ and $\mathsf{J}(\mathsf{r}) = 0$ for $\mathsf{r} \geq 1$
- Critical strain: $\mathsf{S}_\mathsf{c}(\mathbf{y}, \mathbf{x}) = \dfrac{\pm\bar{\mathsf{r}}}{\sqrt{|\mathbf{y} - \mathbf{x}|}}$

Parameters \ Poisson's ratio	$\nu = 0.22$	$\nu = 0.245$
c	4712.4	4712.4
$\bar{\mathsf{C}}$	-1.0623×10^{12}	-1.7349×10^{11}
β	1.7533×10^{8}	1.5647×10^{8}
$\bar{\mathsf{r}}$	5.3402×10^{-5}	5.6529×10^{-5}

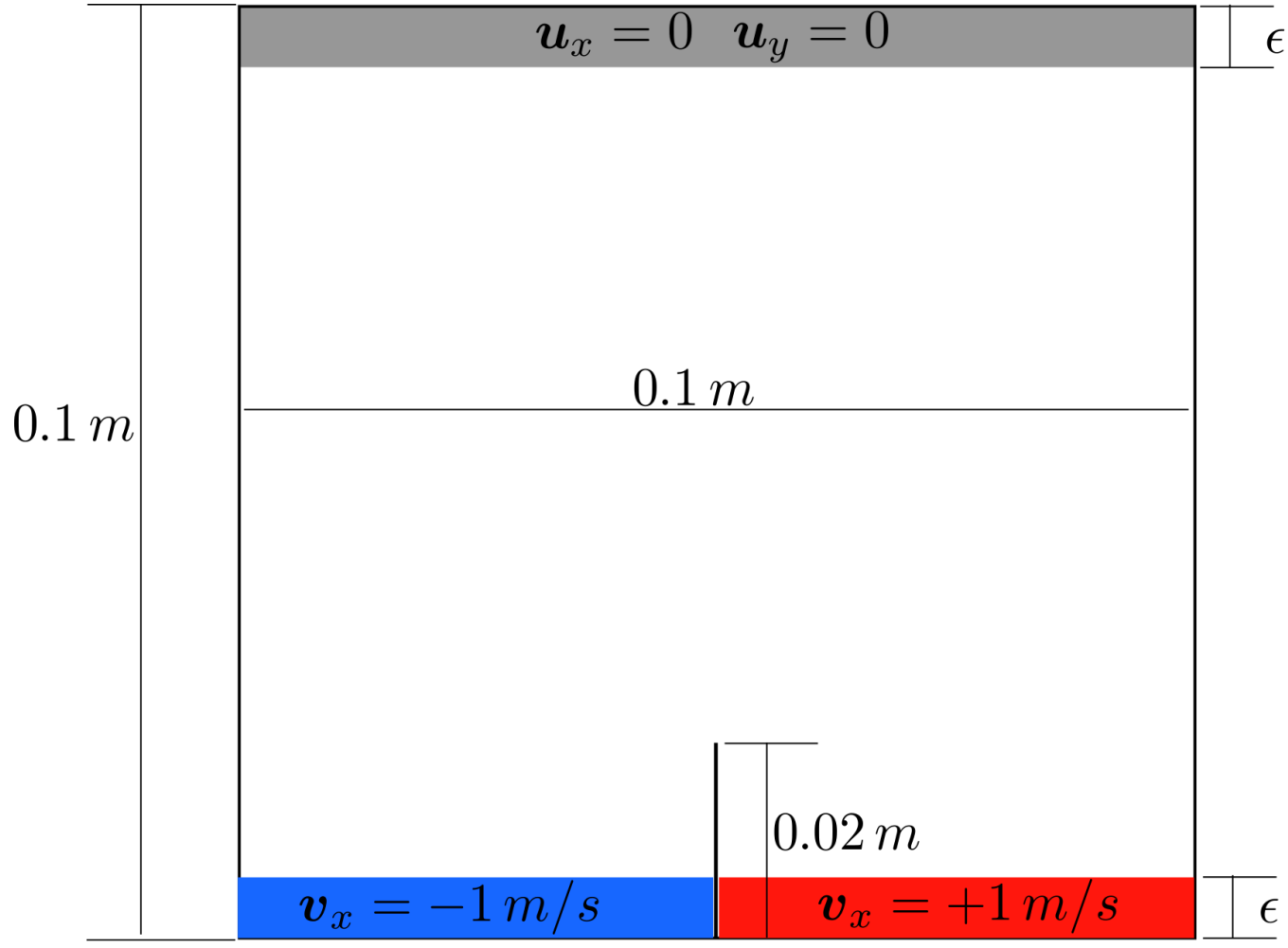
Table 1: Peridynamic material parameters assuming bulk modulus $\mathsf{K} = 25\,\mathsf{GPa}$ and critical energy release rate $\mathsf{G}_\mathsf{c} = 500\,\mathsf{J/m}^{-2}$. Density is $\rho = 1200\,\mathsf{kg/m}^3$. Relation between peridynamic parameters and elastic constant and G_c is established in **Lipton et al (2018)**.

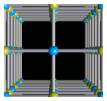
Lipton, Said, & Jha (2018): Dynamic brittle fracture from nonlocal double-well potentials: A state-based model. Handbook of Nonlocal Continuum Mechanics for Materials and Structures, 1–27.



Mode I crack propagation: Setup

- Poisson's ratio: $\nu = 0.245$
- Final time $\mathsf{T} = 34\mu\,\mathsf{s}$, time step $\Delta\mathsf{t} = 0.004\mu\,\mathsf{s}$
- Uniform grid on square domain $\mathsf{D} = [0, 0.1\,\mathsf{m}]^2$

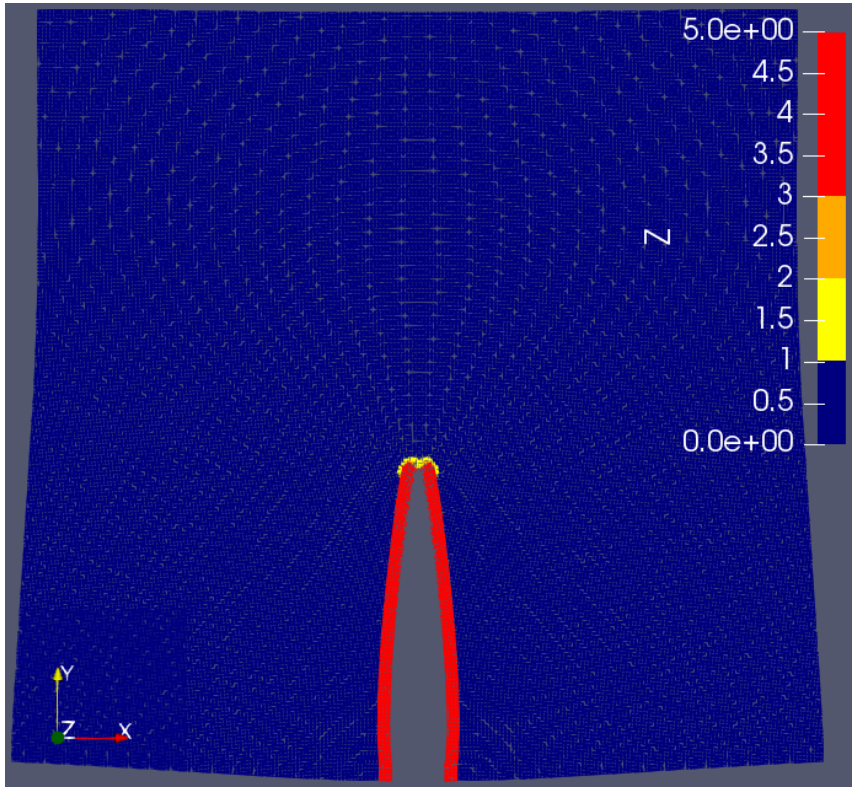


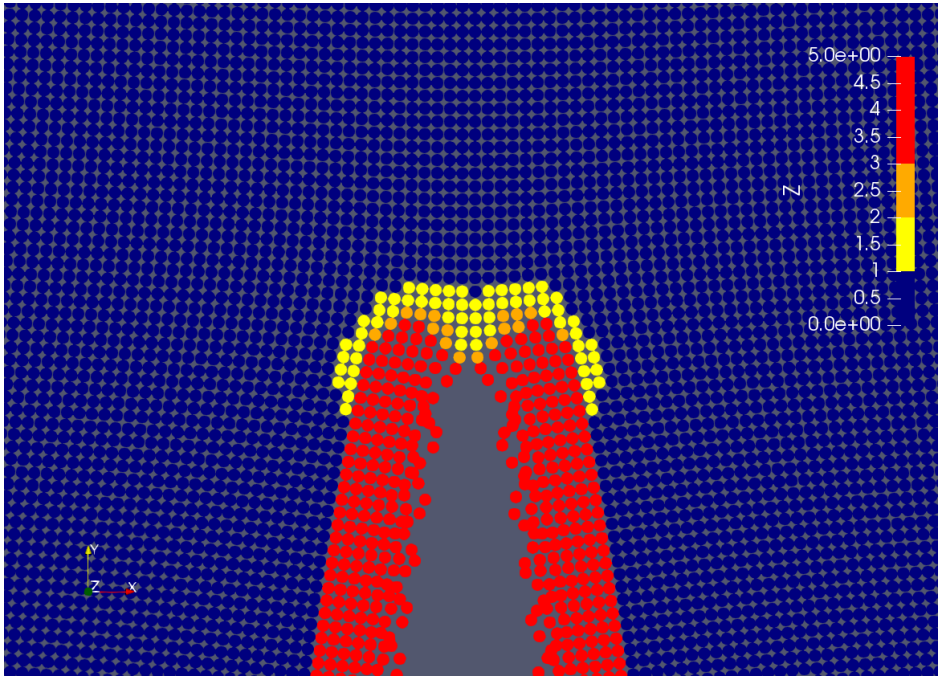


Mode I crack propagation: Damage profile

- Horizon $\epsilon = 4\,\mathsf{mm}$, mesh size $\mathsf{h} = 1\,\mathsf{mm}$
- To measure the extent of damage at material point $\mathbf{x}$ we consider function $Z(\mathbf{x})$ given by

$$Z(\mathbf{x}) = \max_{\mathbf{y} \in H_\epsilon(\mathbf{x}) \cap D} \frac{S(\mathbf{y}, \mathbf{x}; \mathbf{u})}{S_c}$$



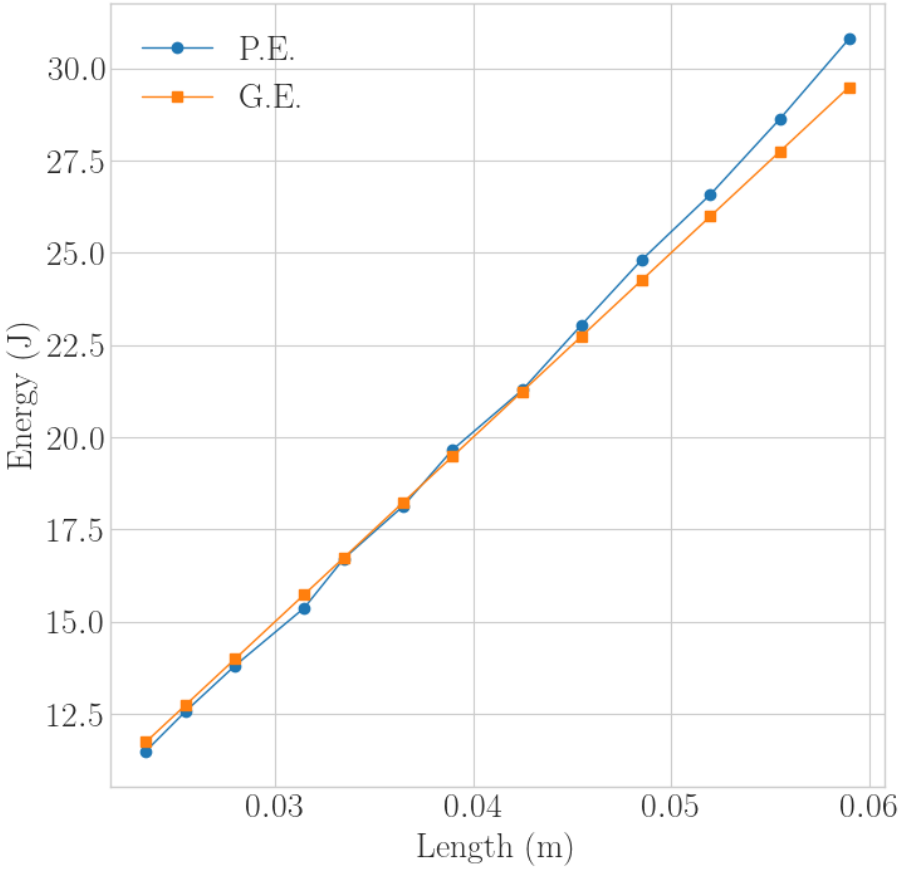


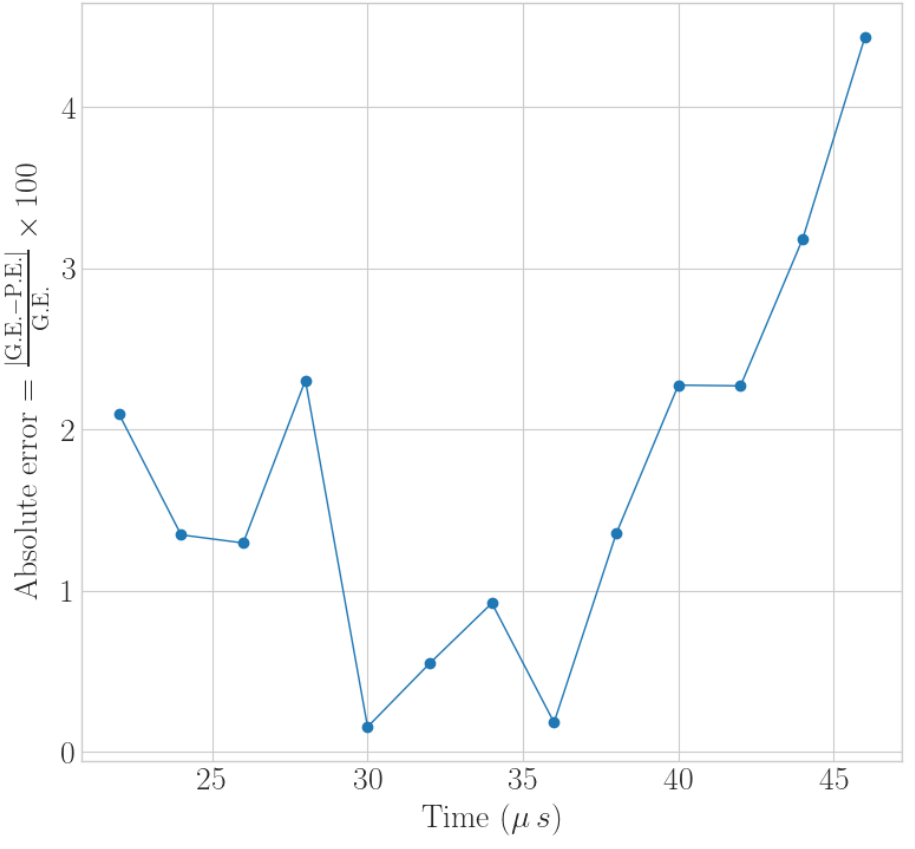
Mode I crack propagation: Fracture energy

- We define the crack zone as set of material points which have $\mathsf{Z} > 1$. For a crack of length l, the Griffith's fracture energy (G.E.) will be $\mathsf{G.E.} = \mathsf{G_c} \times \mathsf{l}$. The peridynamic fracture energy (P.E.) is given by

$$\mathsf{P.E.} = \int_{\substack{\mathbf{x}\in\mathsf{D},\\ \mathsf{Z}(\mathbf{x})\geq 1}} \left[\frac{1}{\epsilon^{\mathsf{d}}\omega_{\mathsf{d}}} \int_{\mathsf{H}_\epsilon(\mathbf{x})} |\mathbf{y}-\mathbf{x}| \mathcal{W}^\epsilon(\mathsf{S}(\mathbf{y},\mathbf{x},\mathbf{u}))\, \mathsf{d}\mathbf{y} \right] \mathsf{d}\mathbf{x},$$

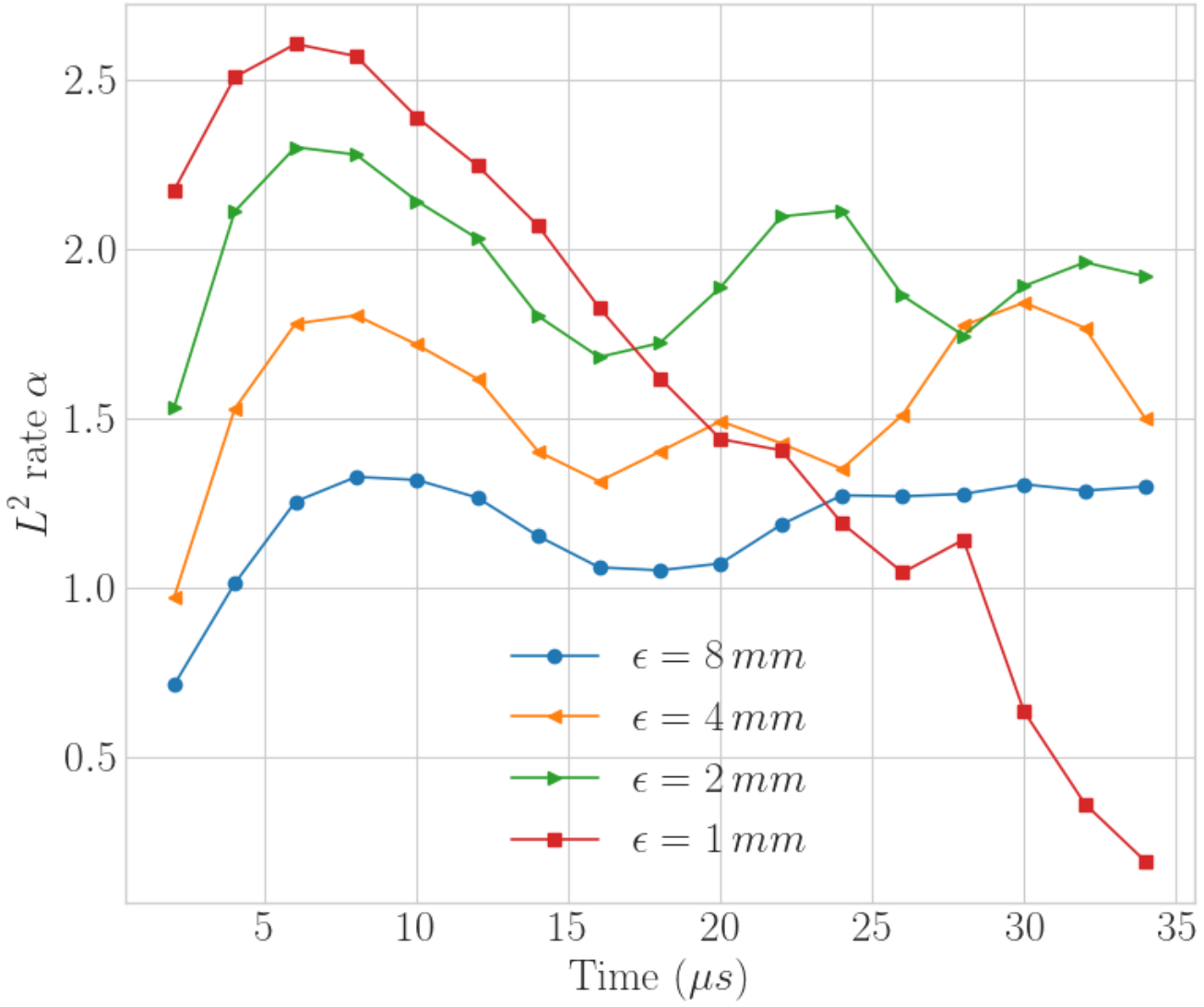
where $\mathcal{W}^\epsilon(\mathsf{S}(\mathbf{y},\mathbf{x},\mathbf{u}))$ is the bond-based potential.

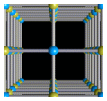




Mode I crack propagation: Convergence rate

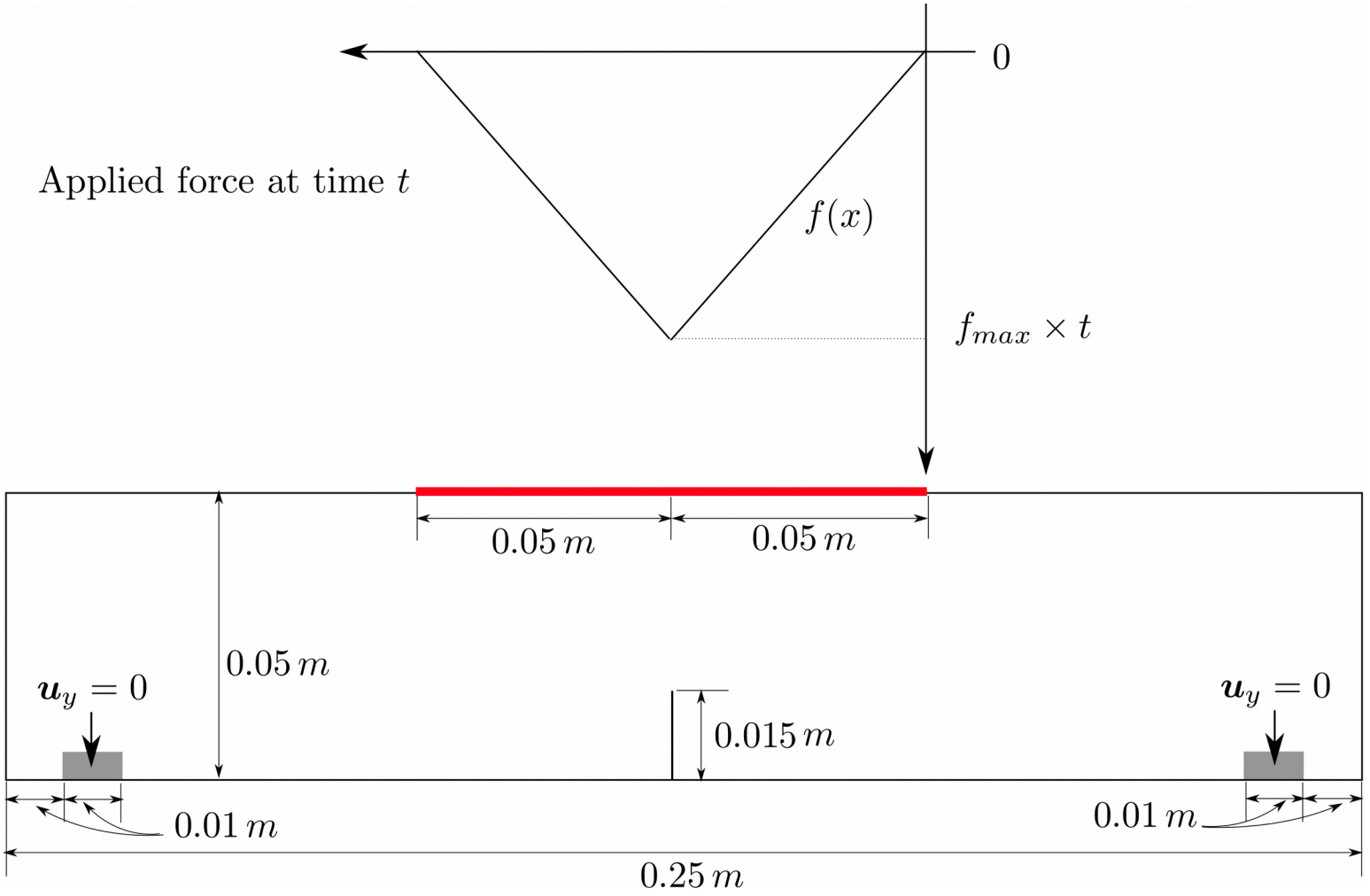
- For each horizon in $8, 4, 2, 1\,\mathsf{mm}$, we obtain numerical results for three mesh sizes $\mathsf{h} = \epsilon/2, \epsilon/4, \epsilon/8$ and compute the rate of convergence



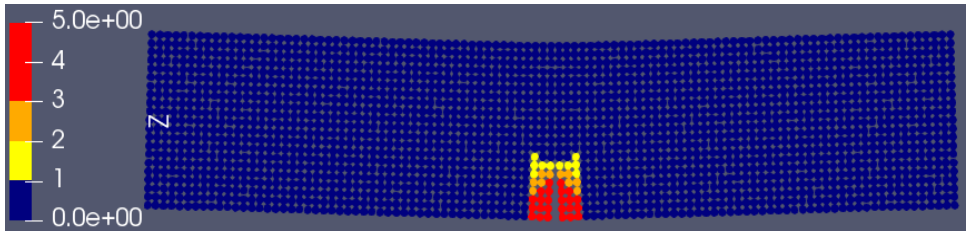


Specimen with pre-crack under bending load

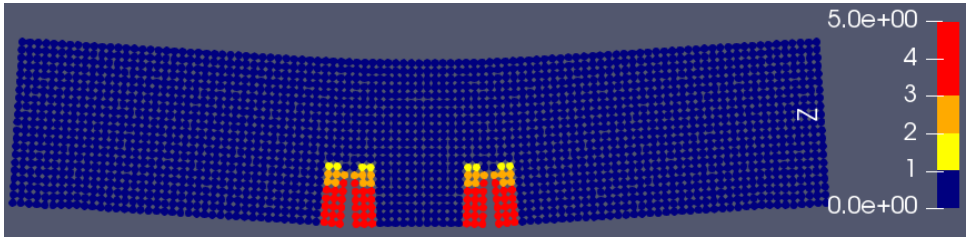
- Poisson's ratio: $\nu = 0.22$
- Final time $\mathsf{T} = 350\mu\,\mathsf{s}$, time step $\Delta\mathsf{t} = 0.0014\mu\,\mathsf{s}$
- Uniform grid on rectangle domain $\mathsf{D} = [0, 0.25\,\mathsf{m}] \times [0, 0.05\,\mathsf{m}]$



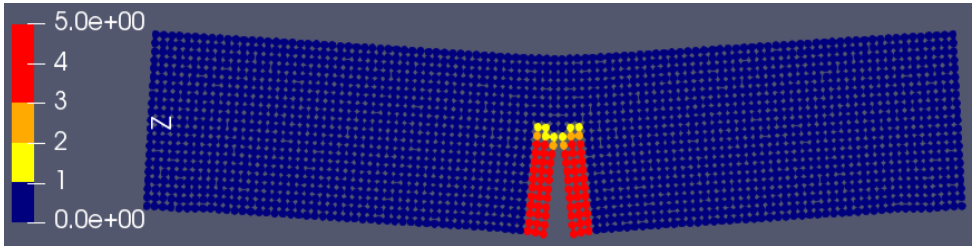
Specimen with pre-crack under bending load



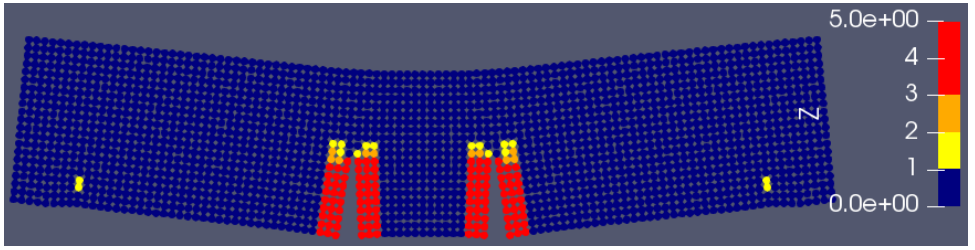
$t = 130\,\mu s$



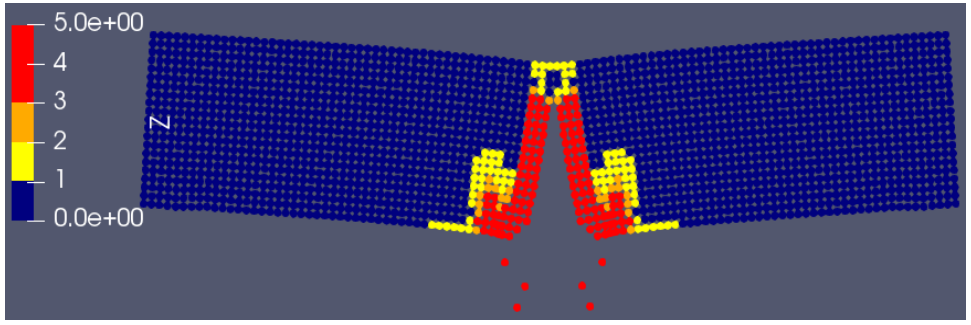
$t = 180\,\mu s$



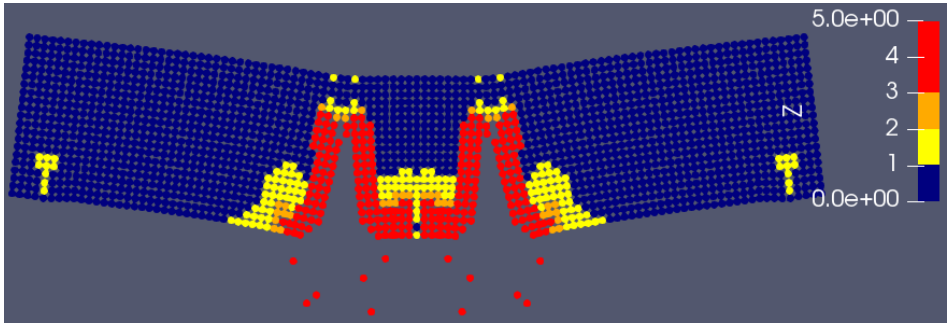
$t = 180\,\mu s$



$t = 220\,\mu s$

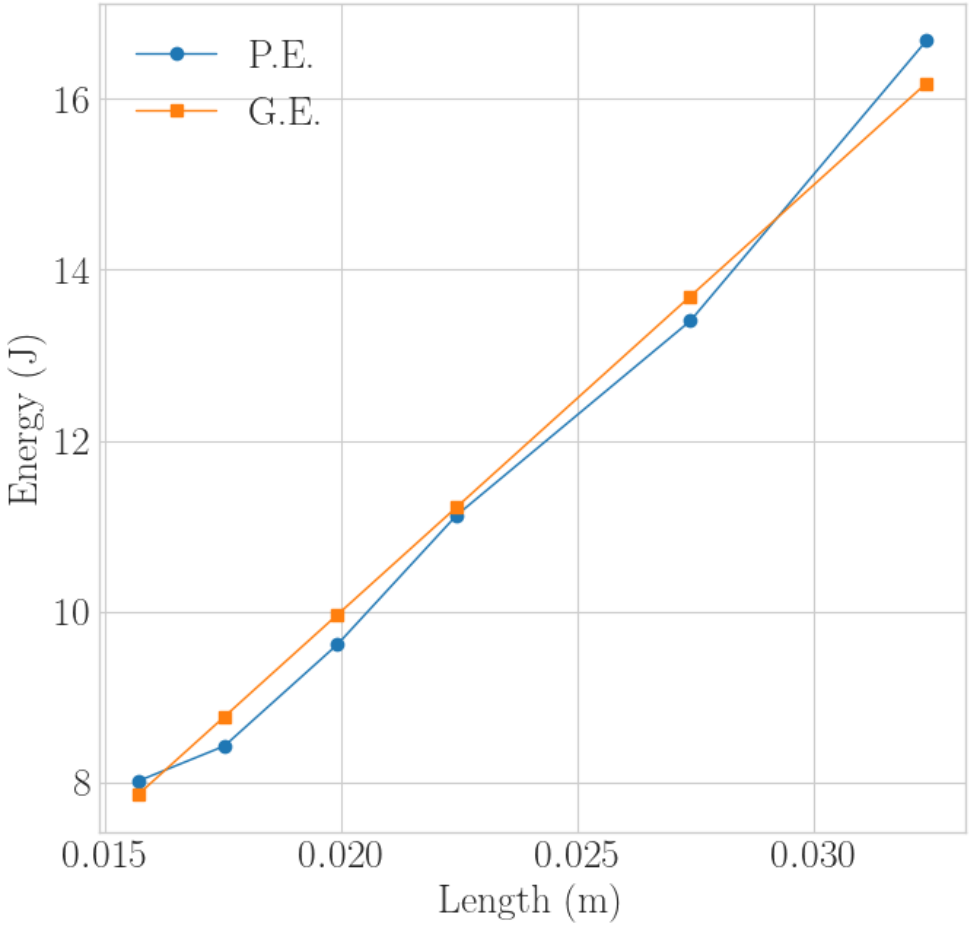


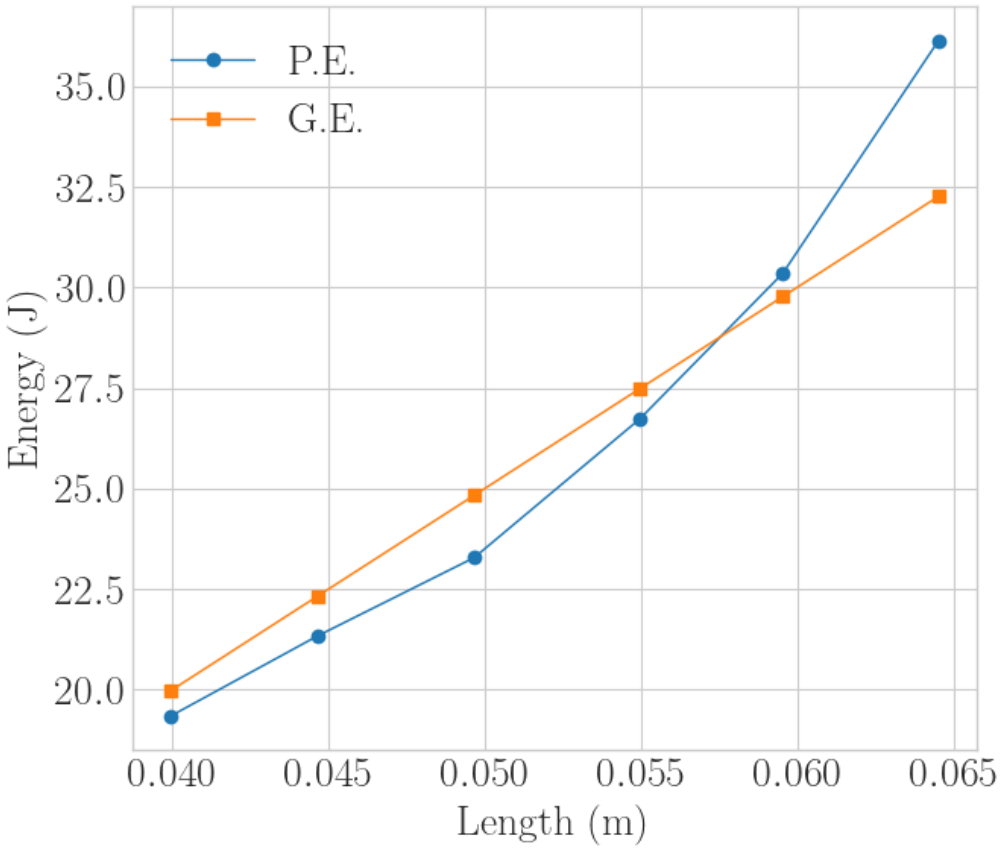
$t = 190\,\mu s$

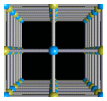


$t = 240\,\mu s$

Specimen with pre-crack under bending load







Finite element approximation

Let V_h is given by linear continuous interpolations over tetrahedral or triangular elements $\mathcal{T}_h$ where h denotes the size of finite element mesh. Here we assume the elements are conforming and the finite element mesh is shape regular and $V_h \subset H_0^1(D;\mathbb{R}^d)$.
For a continuous function $\mathbf{u}$ on $\bar{D}$, $\mathcal{I}_h(\mathbf{u})$ is the continuous piecewise linear interpolant on $\mathcal{T}_h$. It is given by

$$\mathcal{I}_h(\mathbf{u})\Big|_T = \mathcal{I}_T(\mathbf{u}) \qquad \forall T \in \mathcal{T}_h,$$

where $\mathcal{I}_T(\mathbf{u})$ is the local interpolant defined over finite element T and is given by

$$\mathcal{I}_T(\mathbf{u}) = \sum_{i=1}^{n} \mathbf{u}(\mathbf{x}_i)\phi_i.$$

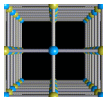
Here n is the number of vertices in an element T, $\mathbf{x}_i$ is the position of vertex i, and ϕ_i is the linear interpolant associated to vertex i.

Let $\mathbf{r}_h(\mathbf{u})$ denote the projection of $\mathbf{u} \in H_0^1(D;\mathbb{R}^d)$ on V_h. For the L^2 norm it is defined as

$$||\mathbf{u} - \mathbf{r}_h(\mathbf{u})|| = \inf_{\tilde{\mathbf{u}} \in V_h} ||\mathbf{u} - \tilde{\mathbf{u}}||.$$

and satisfies

$$(\mathbf{r}_h(\mathbf{u}), \tilde{\mathbf{u}}) = (\mathbf{u}, \tilde{\mathbf{u}}), \qquad \forall \tilde{\mathbf{u}} \in V_h$$



Central difference time discretization

$(\mathbf{u}_h^k, \mathbf{v}_h^k)$ and $(\mathbf{u}^k, \mathbf{v}^k)$ denote the approximate and the exact solution at k^{th} step. Projection is denoted as $(\mathbf{r}_h(\mathbf{u}^k), \mathbf{r}_h(\mathbf{v}^k))$. Approximate initial condition $\mathbf{u}_0, \mathbf{v}_0$ by their projection $\mathbf{r}_h(\mathbf{u}_0), \mathbf{r}_h(\mathbf{v}_0)$ and set $\mathbf{u}_h^0 = \mathbf{r}_h(\mathbf{u}_0)$, $\mathbf{v}_h^0 = \mathbf{r}_h(\mathbf{v}_0)$.

For $k \geq 1$, $(\mathbf{u}_h^k, \mathbf{v}_h^k)$ satisfies, for all $\tilde{\mathbf{u}} \in V_h$,

$$\left(\frac{\mathbf{u}_h^{k+1} - \mathbf{u}_h^k}{\Delta t}, \tilde{\mathbf{u}}\right) = (\mathbf{v}_h^{k+1}, \tilde{\mathbf{u}}),$$

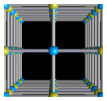
$$\left(\frac{\mathbf{v}_h^{k+1} - \mathbf{v}_h^k}{\Delta t}, \tilde{\mathbf{u}}\right) = (\mathbf{F}^\epsilon(t^k), \tilde{\mathbf{u}}) + (\mathbf{b}_h^k, \tilde{\mathbf{u}}),$$

where we denote projection of $\mathbf{b}(t^k)$, $\mathbf{r}_h(\mathbf{b}(t^k))$, as $\mathbf{b}_h^k$. Combining the two equations delivers central difference equation for $\mathbf{u}_h^k$. We have

$$\left(\frac{\mathbf{u}_h^{k+1} - 2\mathbf{u}_h^k + \mathbf{u}_h^{k-1}}{\Delta t^2}, \tilde{\mathbf{u}}\right) = (\mathbf{F}^\epsilon(t^k), \tilde{\mathbf{u}}) + (\mathbf{b}_h^k, \tilde{\mathbf{u}}), \qquad \forall \tilde{\mathbf{u}} \in V_h.$$

For $k = 0$, we have $\forall \tilde{\mathbf{u}} \in V_h$

$$\left(\frac{\mathbf{u}_h^1 - \mathbf{u}_h^0}{\Delta t^2}, \tilde{\mathbf{u}}\right) = \frac{1}{2}(\mathbf{f}^\epsilon(\mathbf{u}_h^0), \tilde{\mathbf{u}}) + \frac{1}{\Delta t}(\mathbf{v}_h^0, \tilde{\mathbf{u}}) + \frac{1}{2}(\mathbf{b}_h^0, \tilde{\mathbf{u}}).$$



A priori convergence

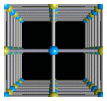
- Well-posedness of solutions in $H^2(D;\mathbb{R}^d) \cap H_0^1(D;\mathbb{R}^d)$ space and existence of $C^2([0,T];H^2(D;\mathbb{R}^d) \cap H_0^1(D;\mathbb{R}^d))$ is shown in **Jha & Lipton (2018b)**
- $E^k := ||\mathbf{u}_h^k - \mathbf{u}(t^k)|| + ||\mathbf{v}_h^k - \mathbf{v}(t^k)||$

Theorem 2. *Let $\epsilon > 0$ be fixed. Let $(\mathbf{u}, \mathbf{v})$ be the solution of peridynamic equation. We assume $\mathbf{u}, \mathbf{v} \in C^2([0,T];H^2(D;\mathbb{R}^d) \cap H_0^1(D;\mathbb{R}^d))$. Then the finite difference scheme is consistent in both time and spatial discretization and converges to the exact solution uniformly in time with respect to the L^2 norm. If we assume the error at the initial step is zero then the error E^k at time t^k is bounded and satisfies*

$$\sup_{0 \le k \le T/\Delta t} E^k \le O\left(C_t \Delta t + C_s \frac{h^2}{\epsilon^2}\right),$$

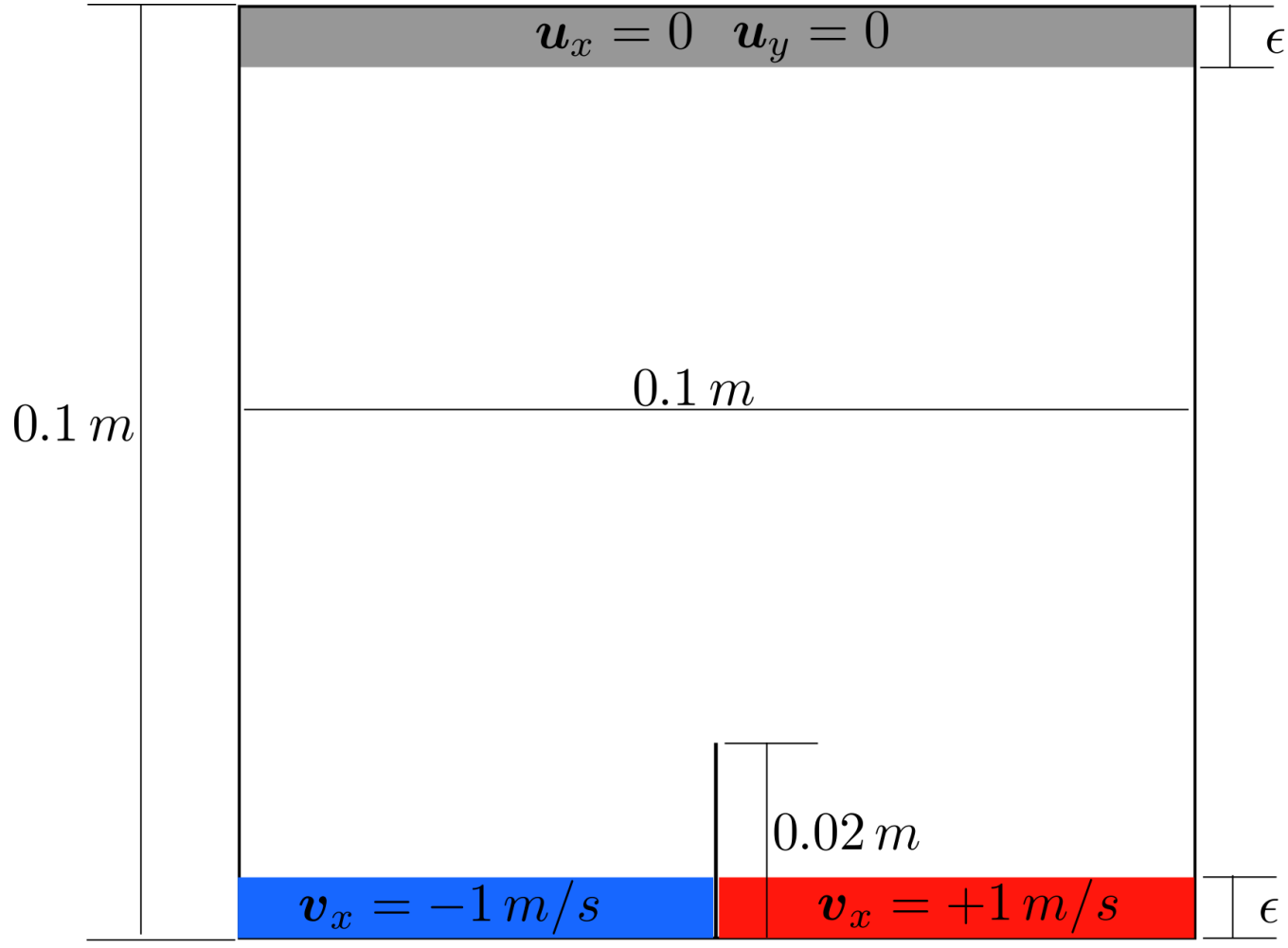
where constant C_s and C_t are independent of h and Δt and C_s depends on the H^2 norm of the solution and C_t depends on the L^2 norms of time derivatives of the solution.

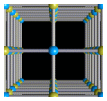
Jha & Lipton (2018b): Finite element convergence for state-based peridynamic fracture models. Currently under review in Communications in Applied Mathematics and Computation (CAMC).



Mode I crack propagation: Setup

- Poisson's ratio: $\nu = 0.245$
- Final time $\mathsf{T} = 40\mu\,\mathsf{s}$, time step $\Delta\mathsf{t} = 0.004\mu\,\mathsf{s}$
- Uniform grid on square domain $\mathsf{D} = [0, 0.1\,\mathsf{m}]^2$

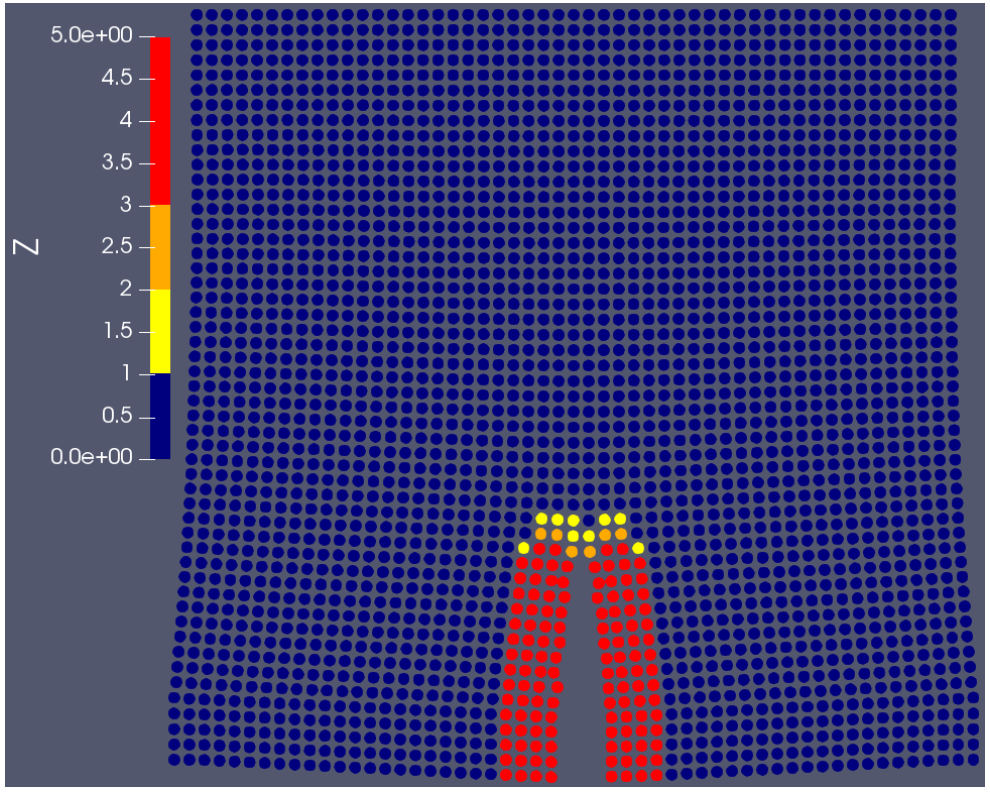




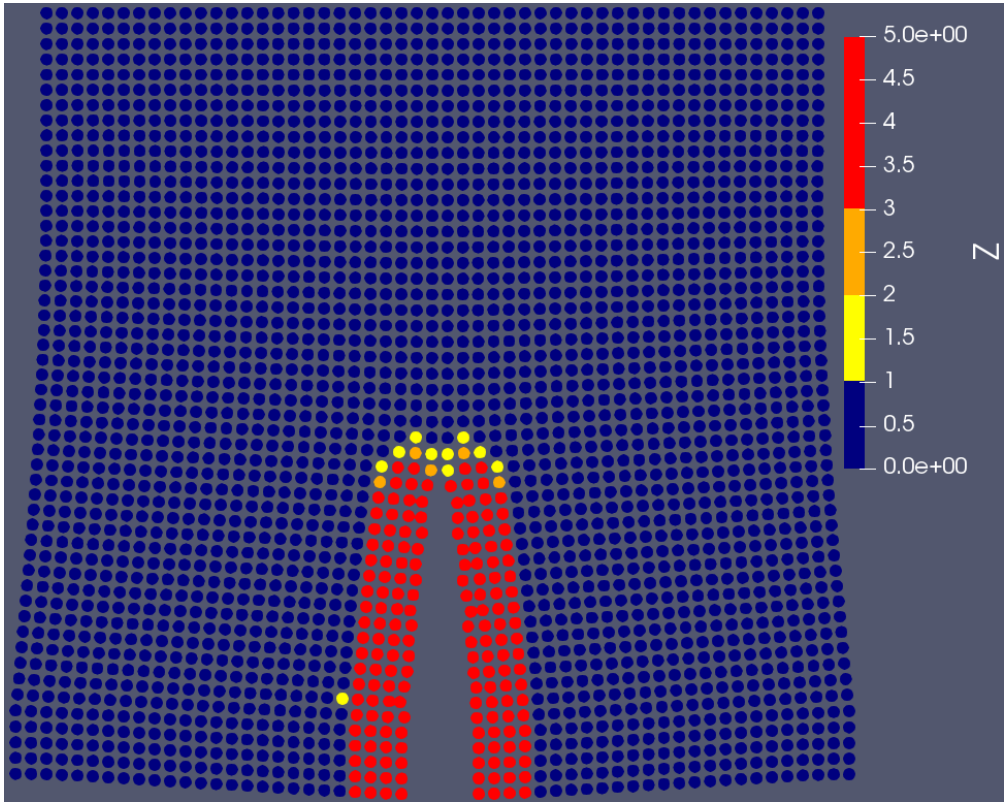
Mode I crack propagation: Damage profile

- Horizon $\epsilon = 8\,\mathsf{mm}$, mesh size $\mathsf{h} = 2\,\mathsf{mm}$
- To measure the extent of damage at material point $\mathbf{x}$ we consider function $\mathsf{Z}(\mathbf{x})$ given by

$$\mathsf{Z}(\mathbf{x}) = \max_{\mathbf{y} \in \mathsf{H}_\epsilon(\mathbf{x}) \cap \mathsf{D}} \frac{\mathsf{S}(\mathbf{y}, \mathbf{x}; \mathbf{u})}{\mathsf{S}_c}$$

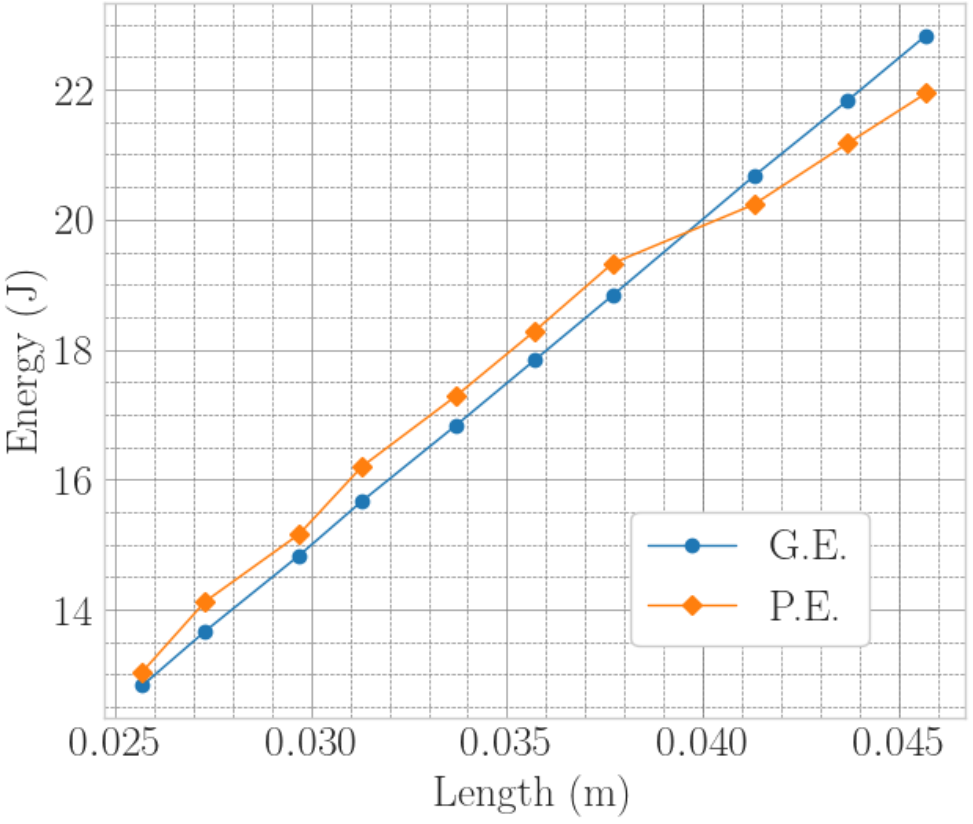


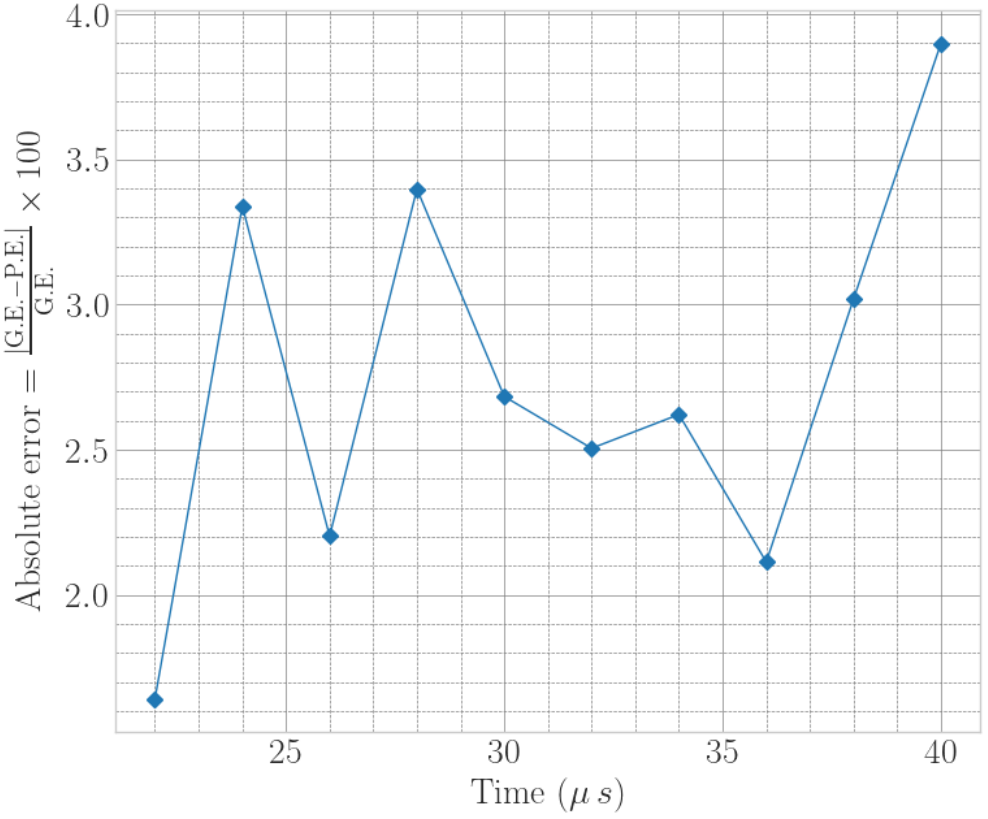
$\mathsf{t} = 30\,\mu\,\mathsf{s}$



$\mathsf{t} = 40\,\mu\,\mathsf{s}$

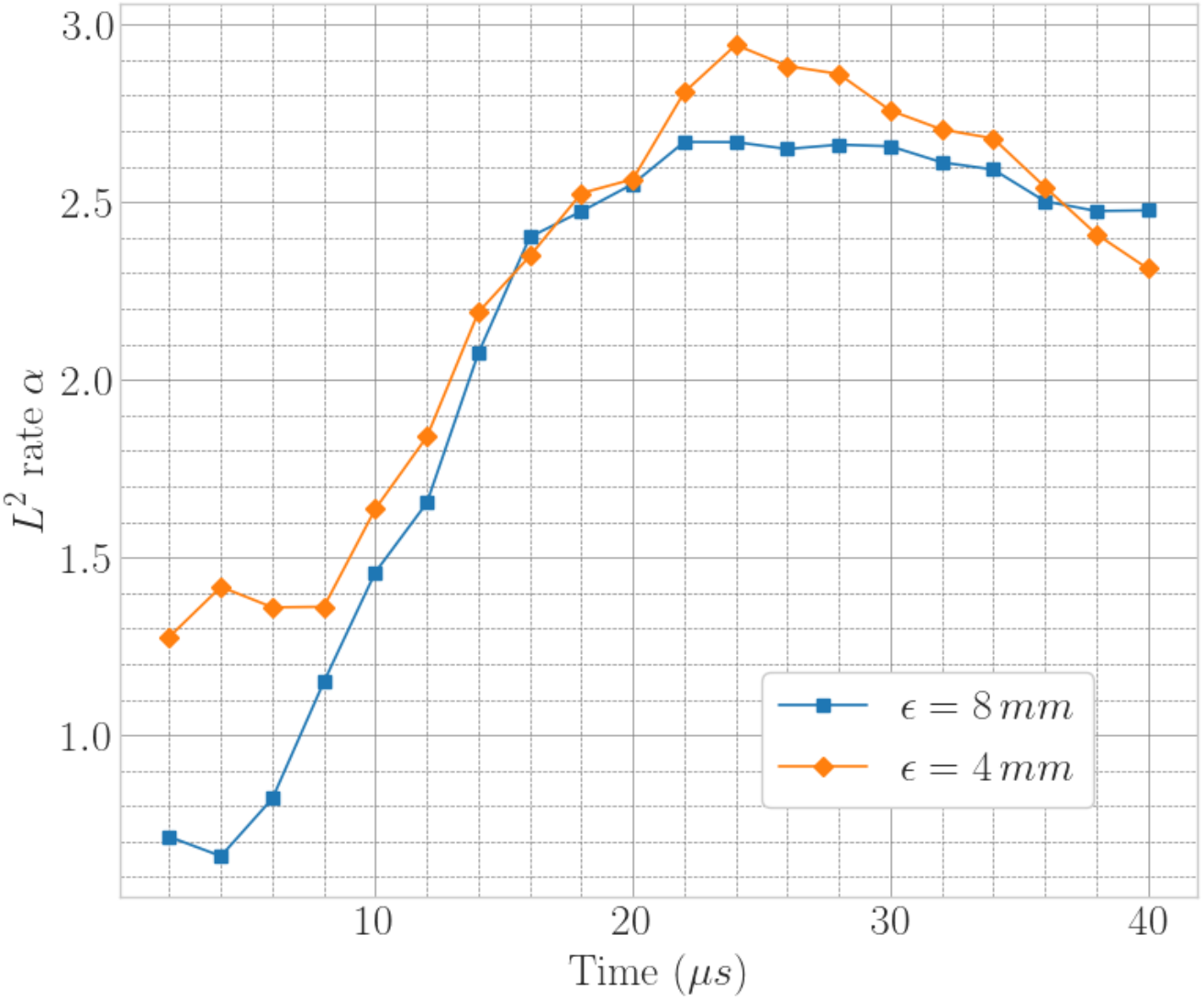
Mode I crack propagation: Fracture energy

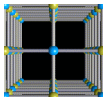




Mode I crack propagation: Convergence rate

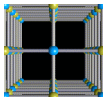
- For each 8 mm and 4 mm horizon, we obtain numerical results for three mesh sizes $h = 2, 1, 0.5$ mm and compute the rate of convergence





Summary

- Numerical analysis of state-based peridynamics model.
- Well-posedness of evolutions in appropriate spaces for error estimates.
- Implementation of finite difference and finite element approximation.
- Good agreement between a priori rate and numerical convergence rate for both finite difference and finite element approximation.
- Peridynamic fracture energy is shown to be close to Griffith's fracture energy for mode I crack propagation.



Future works

- Comparison between peridynamics model (in Quasistatic setting) and phase field models.
- Coupling of local and nonlocal models to reduce computational time and memory consumption.
- 3-d implementation of peridynamic code.

Thank you!