Finite element approximation of nonlocal fracture models

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Outline of talk

- Peridynamic: Introduction
- Well-posedness of Peridynamic solutions
- A priori error estimates on finite element approximations
- Numerical verification
- Future works

Introduction

Let D be the material domain, $\mathsf{D_c}$ be nonlocal boundary, and $\mathbf{u}$ be the displacement field.

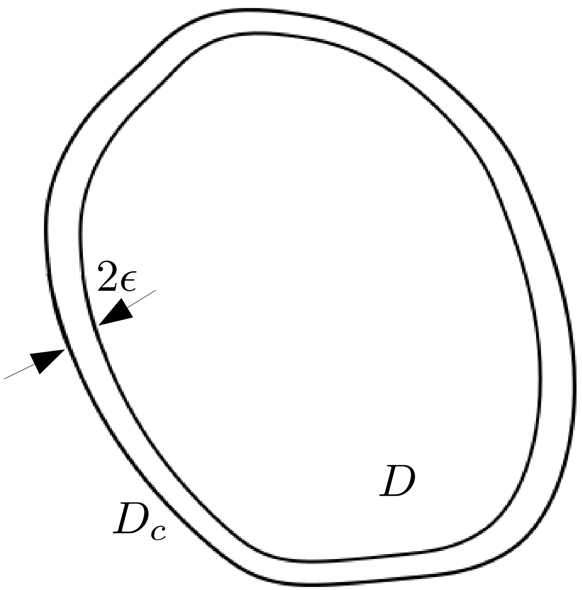
Let $\mathbf{x}$ denote the material point and $\chi(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x})$ is the deformed position.

Strain between two material point $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\mathsf{S}(\mathbf{y}, \mathbf{x}; \mathbf{u}) = \frac{|\mathbf{y} + \mathbf{u}(\mathbf{y}) - \mathbf{x} - \mathbf{u}(\mathbf{x})| - |\mathbf{y} - \mathbf{x}|}{|\mathbf{y} - \mathbf{x}|}$$

Assuming that displacement is small compared to the size of material, we linearize S and get

$$\mathsf{S}(\mathbf{y}, \mathbf{x}; \mathbf{u}) = \frac{\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})}{|\mathbf{y} - \mathbf{x}|} \cdot \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|}$$

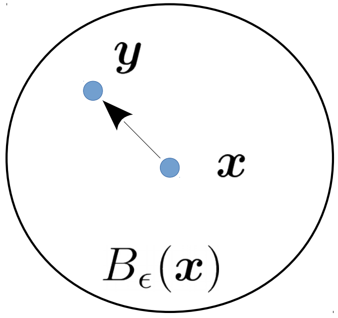


Introduction: Generic force

Consider a material point $\mathbf{x}$. We introduce a length scale ϵ which is called size of horizon. This controls the extent of nonlocal interaction in the material.

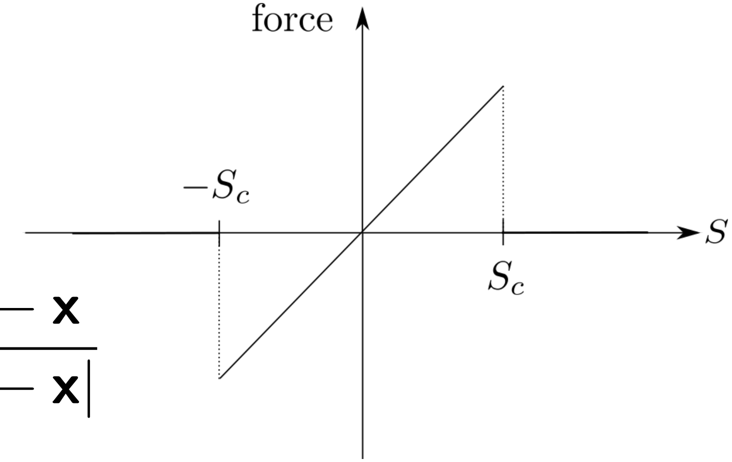
Generic form of force at $\mathbf{x}$ in peridynamic model is given by

$$\mathbf{f}^\epsilon(\mathbf{x};\mathbf{u}) = \frac{1}{|B_\epsilon(\mathbf{x})|}\int_{B_\epsilon(\mathbf{x})} \hat{\mathbf{f}}^\epsilon(\mathbf{y},\mathbf{x};\mathbf{u})d\mathbf{y}$$



$\hat{\mathbf{f}}^\epsilon$ depends on choice of ϵ.

Introduction: Example of a bond-force



- **Example:**

$$\hat{\mathbf{f}}^{\epsilon}(\mathbf{y},\mathbf{x},\mathbf{u}) = \mu(S(\mathbf{y},\mathbf{x};\mathbf{u}))4\frac{J^{\epsilon}(|\mathbf{y}-\mathbf{x}|)}{\epsilon}S(\mathbf{y},\mathbf{x};\mathbf{u})\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}$$

where $\mu(\mathsf{S}) = 1$ if $|\mathsf{S}| < \mathsf{S_c}$ and $\mu(\mathsf{S}) = 0$ when $|\mathsf{S}| \geq \mathsf{S_c}$.

- If $\mathbf{u} \in C^3(D;\mathbb{R}^d)$, and $\sup_{\mathbf{x}\in D} |\nabla^3\mathbf{u}(\mathbf{x})| < \infty$ then

$$\sup_{x\in D} |\mathbf{f}^{\epsilon}(\mathbf{x};\mathbf{u}) - \nabla\cdot\bar{\mathbb{C}}\mathcal{E}\mathbf{u}(\mathbf{x})| = O(\epsilon^2), \quad \bar{\mathbb{C}} = \frac{2}{|B_1(\mathbf{0})|}\int_{B_1(\mathbf{0})} J(|\boldsymbol{\xi}|)\mathbf{e}_{\boldsymbol{\xi}}\otimes\mathbf{e}_{\boldsymbol{\xi}}\otimes\mathbf{e}_{\boldsymbol{\xi}}\otimes\mathbf{e}_{\boldsymbol{\xi}}|\boldsymbol{\xi}|d\boldsymbol{\xi},$$

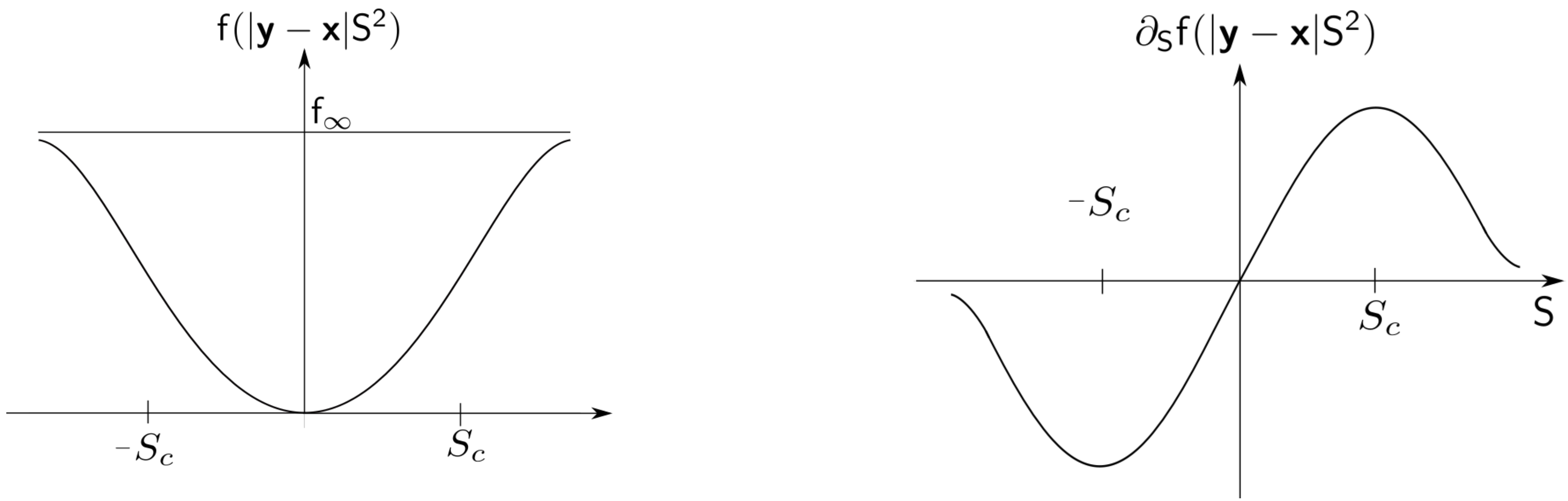
$\mathbf{e}_{\boldsymbol{\xi}} = \boldsymbol{\xi}/|\boldsymbol{\xi}|$ and the strain tensor is $\mathcal{E}\mathbf{u}(\mathbf{x}) = (\nabla\mathbf{u}(\mathbf{x}) + \nabla\mathbf{u}^T(\mathbf{x}))/2$.

Introduction: Regularized force

- We consider peridynamic force of the form

$$\mathbf{f}^\epsilon(\mathbf{x};\mathbf{u}) = \frac{2}{|B_\epsilon(\mathbf{0})|}\int_{B_\epsilon(\mathbf{x})} \frac{J^\epsilon(|\mathbf{y}-\mathbf{x}|)}{\epsilon|\mathbf{y}-\mathbf{x}|}\partial_S f(|\mathbf{y}-\mathbf{x}|S^2)\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}d\mathbf{y}$$

where f is smooth, bounded far away, and linear near origin (**Lipton 2014**[1])



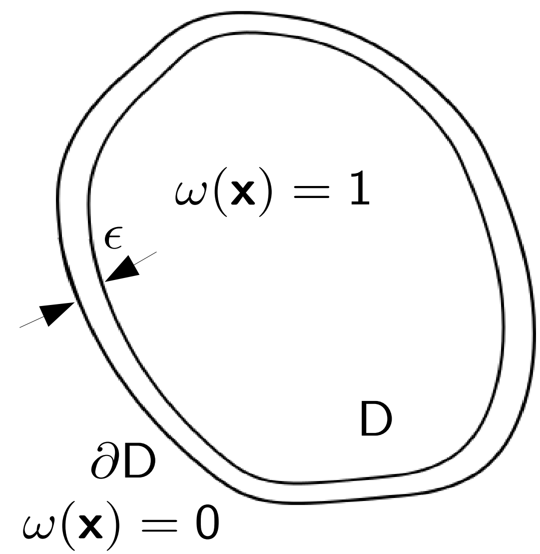
- Critical strain: $S_c(\mathbf{y},\mathbf{x}) = \frac{\bar{r}}{\sqrt{|\mathbf{y}-\mathbf{x}|}}$

[1] Robert Lipton (2014) Dynamic brittle fracture as a small horizon limit of peridynamics Journal of Elasticity 117 no. 1 2150

Introduction: Regularized force

- In **Jha & Lipton 2017a**[1] and **Jha & Lipton 2017b**[2], we introduce boundary function ω in peridynamic force as follows

$$\mathbf{f}^\epsilon(\mathbf{x};\mathbf{u}) = \frac{2}{|B_\epsilon(\mathbf{0})|}\int_{B_\epsilon(\mathbf{x})} \omega(\mathbf{x})\omega(\mathbf{y})\frac{J^\epsilon(|\mathbf{y}-\mathbf{x}|)}{\epsilon|\mathbf{y}-\mathbf{x}|}\partial_S f(|\mathbf{y}-\mathbf{x}|S^2)\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}d\mathbf{y}$$



- With boundary function ω, we can show existence of solutions in regular spaces like $C_0^{0,\gamma}(D;\mathbb{R}^d)$ and $H_0^2(D;\mathbb{R}^d)\cap L^\infty(D;\mathbb{R}^d)$ for Dirichlet boundary condition $\mathbf{u}=\mathbf{0}$ on ∂D.

[1] Prashant K. Jha and Robert Lipton (2018) Numerical analysis of peridynamic models in Hölder space. SIAM Journal on Numerical Analysis

[2] Prashant K. Jha and Robert Lipton (2017) Finite element approximation of nonlocal fracture models. Under review in IMA Journal of Numerical Analysis. arXiv preprint arXiv:1710.07661

Introduction: Equation of motion

- Peridynamics equation: for $\mathbf{x} \in D$ and $t \in [0, T]$

$$\rho \ddot{\mathbf{u}}(\mathbf{x}, t) = \mathbf{f}^\epsilon(\mathbf{x}; \mathbf{u}(t)) + \mathbf{b}(\mathbf{x}, t)$$

- Boundary condition: $\mathbf{u}(\mathbf{x}, t) = \mathbf{0}$ for $\mathbf{x} \in \partial D$ and for $t \in [0, T]$

- Initial condition: $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ and $\dot{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})$ for $\mathbf{x} \in D$

- Weak form: Multiplying peridynamic equation by smooth test function $\tilde{\mathbf{u}}$ such that $\tilde{\mathbf{u}} = \mathbf{0}$ on ∂D, integrating over D, and using nonlocal integration by parts, gives

$$(\rho \ddot{\mathbf{u}}(t), \tilde{\mathbf{u}}) + a^\epsilon(\mathbf{u}(t), \tilde{\mathbf{u}}) = (\mathbf{b}(t), \tilde{\mathbf{u}})$$

where

$$a^\epsilon(\mathbf{u}, \mathbf{v}) = \frac{2}{\epsilon |B_\epsilon(\mathbf{x})|} \int_D \int_{B_\epsilon(x)} \omega(\mathbf{x})\omega(\mathbf{y}) J^\epsilon(|\mathbf{y} - \mathbf{x}|) f'(|\mathbf{y} - \mathbf{x}| S(\mathbf{u})^2) |\mathbf{y} - \mathbf{x}| S(\mathbf{u}) S(\mathbf{v}) d\mathbf{y} d\mathbf{x}$$

and

$$S(\mathbf{u}) = \frac{\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})}{|\mathbf{y} - \mathbf{x}|} \cdot \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|}, S(\mathbf{v}) = \frac{\mathbf{v}(\mathbf{y}) - \mathbf{v}(\mathbf{x})}{|\mathbf{y} - \mathbf{x}|} \cdot \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|}$$

Finite element approximation

We approximate peridynamic equation using linear continuous finite elements. We focus on following three key points

- Well-posedness of peridynamic equation in $H^2_0(D; \mathbb{R}^d)$ space.
- Apriori error estimates due to finite element approximations for exact solutions in $H^2_0(D; \mathbb{R}^d)$.
- Numerical verifications of convergence rate.

Well-posedness of peridynamic equation

Let W denote the $H^2_0(D; \mathbb{R}^d) \cap L^\infty(D; \mathbb{R}^d)$ space. Norm on W is defined as

$$||\mathbf{u}||_W := ||\mathbf{u}||_2 + ||\mathbf{u}||_\infty$$

We will assume that $\mathbf{u} \in H^2_0(D; \mathbb{R}^d)$ is extended by zero outside D, therefore, $\mathbf{u} = \mathbf{0}, \nabla \mathbf{u} = \mathbf{0}, \nabla^2 \mathbf{u} = \mathbf{0}$ for $\mathbf{x} \notin D$ and $||\mathbf{u}||_{H^2(D;\mathbb{R}^d)} = ||\mathbf{u}||_{H^2(\mathbb{R}^d;\mathbb{R}^d)}$.

To show existence of solutions in W, we proceed as follows:

- Obtain Lipschitz bound on peridynamic force in W.
- Using Lipshitz bound, show local existence of unique solutions.

Show that local existence of unique solutions can be repeatedly applied to get global existence of solutions for any time domain $(-T, T)$.

Well-posedness of peridynamic equation

We write the peridynamics equation as an equivalent first order system with $y_1(t) = \mathbf{u}(t)$ and $y_2(t) = \mathbf{v}(t)$ with $\mathbf{v}(t) = \dot{\mathbf{u}}(t)$. Let $y = (y_1, y_2)^T$ where $y_1, y_2 \in W$ and let $F^\epsilon(y,t) = (F_1^\epsilon(y,t), F_2^\epsilon(y,t))^T$ such that

$$F_1^\epsilon(y,t) := y_2,$$

$$F_2^\epsilon(y,t) := \mathbf{f}^\epsilon(y_1) + \mathbf{b}(t).$$

The second order boundary value problem is equivalent to the system of two first order boundary value problem given by

$$\dot{y}(t) = F^\epsilon(y,t),$$

with initial condition given by $y(0) = (\mathbf{u}_0, \mathbf{v}_0)^T \in W \times W$.

Lipschitz bound on peridynamic force

Theorem 1. ***Lipschitz bound on peridynamics force***
Assuming $|\nabla \omega| \leq C_{\omega_1} < \infty$, $|\nabla^2 \omega| \leq C_{\omega_2} < \infty$, *for any* $\mathbf{u}, \mathbf{v} \in W$, *we have*

$$||\mathbf{f}^\epsilon(\mathbf{u}) - \mathbf{f}^\epsilon(\mathbf{v})||_W \leq \frac{\bar{L}_1 + \bar{L}_2(||\mathbf{u}||_W + ||\mathbf{v}||_W) + \bar{L}_3(||\mathbf{u}||_W + ||\mathbf{v}||_W)^2}{\epsilon^3} ||\mathbf{u} - \mathbf{v}||_W$$

where constants $\bar{L}_1, \bar{L}_2, \bar{L}_3$ *are independent of* ϵ, $\mathbf{u}$, *and* $\mathbf{v}$. *Also, for* $\mathbf{u} \in W$, *we have*

$$||\mathbf{f}^\epsilon(\mathbf{u})||_W \leq \frac{\bar{L}_4 ||\mathbf{u}||_W + \bar{L}_5 ||\mathbf{u}||_W^2}{\epsilon^{5/2}},$$

where constants are independent of ϵ *and* $\mathbf{u}$.

Local existence

Theorem 2. ***Local existence and uniqueness***
Given $\mathsf{X} = \mathsf{W} \times \mathsf{W}$, $\mathbf{b}(\mathsf{t}) \in \mathsf{W}$, *and initial data* $\mathsf{x}_0 = (\mathbf{u}_0, \mathbf{v}_0) \in \mathsf{X}$. *We suppose that* $\mathbf{b}(\mathsf{t})$ *is continuous in time over some time interval* $\mathsf{I}_0 = (-\mathsf{T}, \mathsf{T})$ *and satisfies* $\sup_{\mathsf{t} \in \mathsf{I}_0} ||\mathbf{b}(\mathsf{t})||_{\mathsf{W}} < \infty$. *Then, there exists a time interval* $\mathsf{I}' = (-\mathsf{T}', \mathsf{T}') \subset \mathsf{I}_0$ *and unique solution* $\mathsf{y} = (\mathsf{y}^1, \mathsf{y}^2)$ *such that* $\mathsf{y} \in \mathsf{C}^1(\mathsf{I}'; \mathsf{X})$ *and*

$$y(t) = x_0 + \int_0^t F^{\epsilon}(y(\tau), \tau)\, d\tau, \ \textit{for } \mathsf{t} \in \mathsf{I}'$$

or equivalently

$$y'(t) = F^{\epsilon}(y(t), t), \textit{with } \mathsf{y}(0) = \mathsf{x}_0, \ \textit{for } \mathsf{t} \in \mathsf{I}'$$

where $\mathsf{y}(\mathsf{t})$ *and* $\mathsf{y}'(\mathsf{t})$ *are Lipschitz continuous in time for* $\mathsf{t} \in \mathsf{I}' \subset \mathsf{I}_0$.

Local existence

Proof: Let $T' > 0$ and $Y(T')$ be a set of functions $y(t) \in W$ for $t \in (-T', T')$. We show that there exists such a set $Y(T')$ and $T' > 0$ such that map S_{x_0}, defined as follows

$$S_{x_0}(y)(t) = x_0 + \int_0^t F^{\epsilon}(y(\tau), \tau)\, d\tau,$$

or in element form

$$S^1_{x_0}(y)(t) = x^1_0 + \int_0^t y^2(\tau)\, d\tau$$

$$S^2_{x_0}(y)(t) = x^2_0 + \int_0^t (\mathbf{f}^{\epsilon}(y^1(\tau)) + \mathbf{b}(\tau))\, d\tau,$$

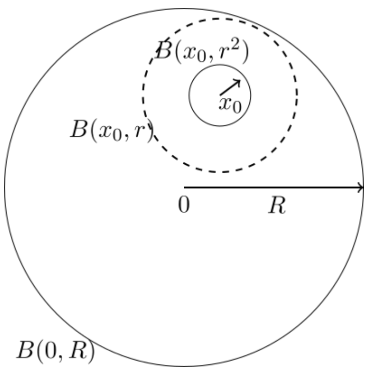
maps functions in $Y(T')$ to functions in $Y(T')$. We then apply fixed point theorem such as in **Driver 2003**[1].

[1] Bruce K Driver (2003) Analysis tools with applications. Lecture notes.

Local existence

Write $y(t) = (y^1(t), y^2(t))^T$ with $||y||_X = ||y^1(t)||_W + ||y^2(t)||_W$. Let $R > ||x_0||_X$ and $B(0,R) = \{y \in X : ||y||_X < R\}$. Let $r < \min\{1, R - ||x_0||_X\}$. We have $r^2 < (R - ||x_0||_X)^2$ and $r^2 < r < R - ||x_0||_X$. Consider the ball $B(x_0, r^2) = \{y \in X : ||y - x_0||_X < r^2\}$.

Then we have $B(x_0, r^2) \subset B(x_0, r) \subset B(0, R)$.



Introduce $0 < T' < T$ and the associated set $Y(T')$ of functions in W taking values in $B(x_0, r^2)$ for $I' = (-T', T') \subset I_0 = (-T, T)$. I.e. for all $y \in Y(T')$, $y(t) \in B(x_0, r^2)$ for all $t \in (-T', T')$. We want to find T' such that $S_{x_0}(y)(t) \in B(x_0, r^2)$ for all $t \in (-T', T')$ implying $S_{x_0}(y) \in Y(T')$.

Writing out the transformation with $y(t) \in Y(T')$ gives

$$S^1_{x_0}(y)(t) = x^1_0 + \int_0^t y^2(\tau)\, d\tau$$

$$S^2_{x_0}(y)(t) = x^2_0 + \int_0^t (\mathbf{f}^\epsilon(y^1(\tau)) + b(\tau))\, d\tau.$$

We simply have

$$||S^1_{x_0}(y)(t) - x^1_0||_W \leq \sup_{t \in (-T', T')} ||y^2(t)||_W T'.$$

Local existence

Using bound on $\mathbf{f}^{\epsilon}$, we have

$$||S^2_{x_0}(y)(t) - x^2_0||_W \leq \int_0^t \left[\frac{\bar{L}_4}{\epsilon^{5/2}} ||y^1(\tau)||_W + \frac{\bar{L}_5}{\epsilon^{5/2}} ||y^1(\tau)||^2_W + ||\mathbf{b}(\tau)||_W \right] d\tau.$$

Let $\bar{b} = \sup_{t \in I_0} ||\mathbf{b}(t)||_W$. Noting that transformation S_{x_0} is defined for $t \in I' = (-T', T')$ and $y(\tau) = (y^1(\tau), y^2(\tau)) \in B(x_0, r^2) \subset B(0, R)$ as $y \in Y(T')$, we have from 3 and 4

$$||S^1_{x_0}(y)(t) - x^1_0||_W \leq RT',$$
$$||S^2_{x_0}(y)(t) - x^2_0||_W \leq \left[\frac{\bar{L}_4 R + \bar{L}_5 R^2}{\epsilon^{5/2}} + \bar{b} \right] T'.$$

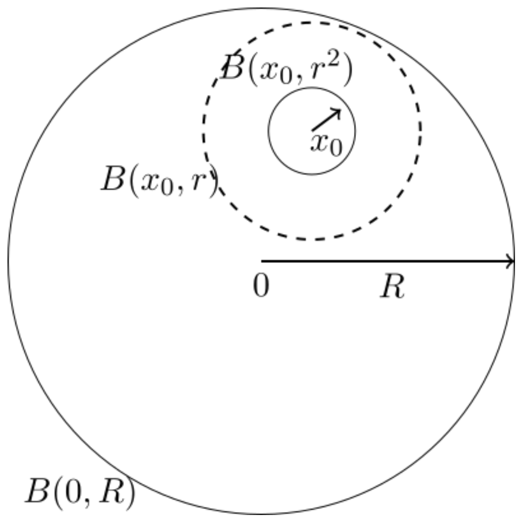
Combining to get

$$||S_{x_0}(y)(t) - x_0||_X \leq \left[\frac{\bar{L}_4 R + \bar{L}_5 R^2}{\epsilon^{5/2}} + R + \bar{b} \right] T'.$$

Local existence

Choosing T' as follow: $\quad \mathsf{T}' < \dfrac{\mathsf{r}^2}{\left[\frac{\bar{\mathsf{L}}_4\mathsf{R}+\bar{\mathsf{L}}_5\mathsf{R}^2}{\epsilon^{5/2}} + \mathsf{R} + \bar{\mathsf{b}}\right]}$

Then $\mathsf{S}_{\mathsf{x}_0}(\mathsf{y}) \in \mathsf{Y}(\mathsf{T}')$ for all $\mathsf{y} \in \mathsf{Y}(\mathsf{T}')$ as $\quad ||\mathsf{S}_{\mathsf{x}_0}(\mathsf{y})(\mathsf{t}) - \mathsf{x}_0||_\mathsf{X} < \mathsf{r}^2$.



Since $\mathsf{r}^2 < (\mathsf{R} - ||\mathsf{x}_0||_\mathsf{X})^2$, we have

$$\mathsf{T}' < \frac{\mathsf{r}^2}{\left[\frac{\bar{\mathsf{L}}_4\mathsf{R}+\bar{\mathsf{L}}_5\mathsf{R}^2}{\epsilon^{5/2}} + \mathsf{R} + \bar{\mathsf{b}}\right]} < \frac{(\mathsf{R} - ||\mathsf{x}_0||_\mathsf{X})^2}{\left[\frac{\bar{\mathsf{L}}_4\mathsf{R}+\bar{\mathsf{L}}_5\mathsf{R}^2}{\epsilon^{5/2}} + \mathsf{R} + \bar{\mathsf{b}}\right]}.$$

Let $\theta(\mathsf{R})$ be given by

$$\theta(R) := \frac{(R - ||x_0||_X)^2}{\left[\frac{\bar{L}_4 R+\bar{L}_5 R^2}{\epsilon^{5/2}} + R + \bar{b}\right]}.$$

Note that $\theta(\mathsf{R})$ is increasing with $\mathsf{R} > 0$ and satisfies

$$\theta_\infty := \lim_{\mathsf{R}\to\infty} \theta(\mathsf{R}) = \frac{\epsilon^{5/2}}{\bar{\mathsf{L}}_5}.$$

Local existence

So given R and $||x_0||_X$ we choose T' according to

$$\frac{\theta(R)}{2} < T' < \theta(R),$$

and set $I' = (-T', T')$. This way we have shown that for time domain I' the transformation $S_{x_0}(y)(t)$ maps $Y(T')$ into itself. Existence and uniqueness of solution can be established using Theorem 6.10 in **Driver 2003**[1].

[1] Bruce K Driver (2003) Analysis tools with applications. Lecture notes.

Global existence

Theorem 3. ***Existence and uniqueness of solutions over finite time intervals***
For any initial condition $\mathsf{x}_0 \in \mathsf{X} = \mathsf{W} \times \mathsf{W}$, *time interval* $\mathsf{I}_0 = (-\mathsf{T}, \mathsf{T})$, *and right hand side* $\mathbf{b}(\mathsf{t})$ *continuous in time for* $\mathsf{t} \in \mathsf{I}_0$ *such that* $\mathbf{b}(\mathsf{t})$ *satisfies* $\sup_{\mathsf{t} \in \mathsf{I}_0} ||\mathbf{b}(\mathsf{t})||_{\mathsf{W}} < \infty$, *there is a unique solution* $\mathsf{y}(\mathsf{t}) \in \mathsf{C}^1(\mathsf{I}_0; \mathsf{X})$ *of*

$$y(t) = x_0 + \int_0^t F^\epsilon(y(\tau), \tau)\, d\tau,$$

or equivalently

$$y'(t) = F^\epsilon(y(t), t), \textit{with } \mathsf{y}(0) = \mathsf{x}_0,$$

where $\mathsf{y}(\mathsf{t})$ *and* $\mathsf{y}'(\mathsf{t})$ *are Lipschitz continuous in time for* $\mathsf{t} \in \mathsf{I}_0$.

Global existence

We have shown a unique local solution over a time domain $(-\mathsf{T}', \mathsf{T}')$ with $\frac{\theta(\mathsf{R})}{2} < \mathsf{T}'$. Since $\theta(\mathsf{R}) \nearrow \epsilon^{5/2}/\bar{\mathsf{L}}_5$ as $\mathsf{R} \nearrow \infty$ we can fix a tolerance $\eta > 0$ so that $[(\epsilon^{5/2}/2\bar{\mathsf{L}}_5) - \eta] > 0$.

Then for any initial condition in W and $\mathsf{b} = \sup_{\mathsf{t}\in[-\mathsf{T},\mathsf{T})} ||\mathbf{b}(\mathsf{t})||_{\mathsf{W}}$ we can choose R sufficiently large so that $||\mathsf{x}_0||_{\mathsf{X}} < \mathsf{R}$ and $0 < (\epsilon^{5/2}/2\bar{\mathsf{L}}_5) - \eta < \mathsf{T}'$.

Since choice of T' is independent of initial condition and R, we can always find local solutions for time intervals $(-\mathsf{T}', \mathsf{T}')$ for T' larger than $[(\epsilon^{5/2}/2\bar{\mathsf{L}}_5) - \eta] > 0$. Therefore we apply the local existence and uniqueness result to uniquely continue local solutions up to an arbitrary time interval $(-\mathsf{T}, \mathsf{T})$.

Finite element approximation

Let V_h be the approximation of $H_0^2(D, \mathbb{R}^d)$ associated to the linear continuous interpolation function over triangulation $\mathcal{T}_h$ where h denotes the size of finite element mesh. Let $\mathcal{I}_h(\mathbf{u})$ be defined as below

$$\mathcal{I}_h(\mathbf{u})(\mathbf{x}) = \sum_{T \in \mathcal{T}_h} \left[\sum_{i \in N_T} \mathbf{u}(\mathbf{x}_i) \phi_i(\mathbf{x}) \right].$$

Assuming that the size of each element in triangulation $\mathcal{T}_h$ is bounded by h, we have (see Theorem 4.6 **Arnold 2011**[1])

$$||\mathbf{u} - \mathcal{I}_h(\mathbf{u})|| \le ch^2 ||\mathbf{u}||_2, \qquad \forall \mathbf{u} \in H_0^2(D; \mathbb{R}^d).$$

Projection of function in FE space:

$$||\mathbf{u} - \mathbf{r}_h(\mathbf{u})|| = \inf_{\tilde{\mathbf{u}} \in V_h} ||\mathbf{u} - \tilde{\mathbf{u}}||.$$

We have

$$(\mathbf{r}_h(\mathbf{u}), \tilde{\mathbf{u}}) = (\mathbf{u}, \tilde{\mathbf{u}}), \qquad \forall \tilde{\mathbf{u}} \in V_h.$$

$$||\mathbf{u} - \mathbf{r}_h(\mathbf{u})|| \le ch^2 ||\mathbf{u}||_2 \qquad \forall \mathbf{u} \in H_0^2(D; \mathbb{R}^d).$$

[1] Douglas N. Arnold (2011) Lecture notes on numerical analysis of partial differential equations. http://www.math.umn.edu/ ~ arnold/8445/notes.pdf

Semi-discrete approximation and stability

Let $\mathbf{u}_h(t) \in V_h$ be the approximation of $\mathbf{u}(t)$ which satisfies following

$$(\ddot{\mathbf{u}}_h, \tilde{\mathbf{u}}) + a^\epsilon(\mathbf{u}_h(t), \tilde{\mathbf{u}}) = (\mathbf{b}(t), \tilde{\mathbf{u}}), \qquad \forall \tilde{\mathbf{u}} \in V_h.$$

We show that the semi-discrete approximation is stable, i.e. energy at time t is bounded by initial energy and work done by the body force.

The total energy $\mathcal{E}^\epsilon(\mathbf{u})(t)$ is given by the sum of kinetic and potential energy given by

$$\mathcal{E}^\epsilon(\mathbf{u})(t) = \frac{1}{2}||\dot{\mathbf{u}}(t)||_{L^2} + PD^\epsilon(\mathbf{u}(t)), PD^\epsilon(\mathbf{u}) = \int_D \left[\frac{1}{|B_\epsilon(\mathbf{x})|}\int_{B_\epsilon(\mathbf{x})} W^\epsilon(S(\mathbf{u}), \mathbf{y}-\mathbf{x})d\mathbf{y}\right] d\mathbf{x},$$

where bond-potential is given by $W^\epsilon(S, \mathbf{y}-\mathbf{x}) = \omega(\mathbf{x})\omega(\mathbf{y})\dfrac{J^\epsilon(|\mathbf{y}-\mathbf{x}|)}{\epsilon} f(|\mathbf{y}-\mathbf{x}|S^2)$.

Theorem 4. ***Stability of semi-discrete approximation***
The semi-discrete scheme is stable and the energy $\mathcal{E}^\epsilon(\mathbf{u}_h)(t)$ satisfies the following bound

$$\mathcal{E}^\epsilon(\mathbf{u}_h)(t) \leq \left[\sqrt{\mathcal{E}^\epsilon(\mathbf{u}_h)(0)} + \int_0^t ||\mathbf{b}(\tau)||d\tau\right]^2.$$

Central difference time discretization

$(\mathbf{u}_h^k, \mathbf{v}_h^k)$ and $(\mathbf{u}^k, \mathbf{v}^k)$ denote the approximate and the exact solution at k^{th} step. Projection is denoted as $(\mathbf{r}_h(\mathbf{u}^k), \mathbf{r}_h(\mathbf{v}^k))$. Approximate initial condition $\mathbf{u}_0, \mathbf{v}_0$ by their projection $\mathbf{r}_h(\mathbf{u}_0), \mathbf{r}_h(\mathbf{v}_0)$ and set $\mathbf{u}_h^0 = \mathbf{r}_h(\mathbf{u}_0)$, $\mathbf{v}_h^0 = \mathbf{r}_h(\mathbf{v}_0)$.

For $k \geq 1$, $(\mathbf{u}_h^k, \mathbf{v}_h^k)$ satisfies, for all $\tilde{\mathbf{u}} \in V_h$,

$$
\begin{aligned}
\left(\frac{\mathbf{u}_h^{k+1} - \mathbf{u}_h^k}{\Delta t}, \tilde{\mathbf{u}} \right) &= (\mathbf{v}_h^{k+1}, \tilde{\mathbf{u}}), \\
\left(\frac{\mathbf{v}_h^{k+1} - \mathbf{v}_h^k}{\Delta t}, \tilde{\mathbf{u}} \right) &= (\mathbf{f}^\epsilon(\mathbf{u}_h^k), \tilde{\mathbf{u}}) + (\mathbf{b}_h^k, \tilde{\mathbf{u}}),
\end{aligned}
$$

where we denote projection of $\mathbf{b}(t^k)$, $\mathbf{r}_h(\mathbf{b}(t^k))$, as $\mathbf{b}_h^k$. Combining the two equations delivers central difference equation for $\mathbf{u}_h^k$. We have

$$
\left(\frac{\mathbf{u}_h^{k+1} - 2\mathbf{u}_h^k + \mathbf{u}_h^{k-1}}{\Delta t^2}, \tilde{\mathbf{u}} \right) = (\mathbf{f}^\epsilon(\mathbf{u}_h^k), \tilde{\mathbf{u}}) + (\mathbf{b}_h^k, \tilde{\mathbf{u}}), \qquad \forall \tilde{\mathbf{u}} \in V_h.
$$

For $k = 0$, we have $\forall \tilde{\mathbf{u}} \in V_h$

$$
\left(\frac{\mathbf{u}_h^1 - \mathbf{u}_h^0}{\Delta t^2}, \tilde{\mathbf{u}} \right) = \frac{1}{2}(\mathbf{f}^\epsilon(\mathbf{u}_h^0), \tilde{\mathbf{u}}) + \frac{1}{\Delta t}(\mathbf{v}_h^0, \tilde{\mathbf{u}}) + \frac{1}{2}(\mathbf{b}_h^0, \tilde{\mathbf{u}}).
$$

Convergence of approximation

Error E^k is given by $\mathsf{E}^\mathsf{k} := ||\mathbf{u}_\mathsf{h}^\mathsf{k} - \mathbf{u}(\mathsf{t}^\mathsf{k})|| + ||\mathbf{v}_\mathsf{h}^\mathsf{k} - \mathbf{v}(\mathsf{t}^\mathsf{k})||$. We split the error as follows

$$\mathsf{E}^\mathsf{k} \le \left(||\mathbf{u}^\mathsf{k} - \mathbf{r}_\mathsf{h}(\mathbf{u}^\mathsf{k})|| + ||\mathbf{v}^\mathsf{k} - \mathbf{r}_\mathsf{h}(\mathbf{v}^\mathsf{k})||\right) + \left(||\mathbf{r}_\mathsf{h}(\mathsf{u}^\mathsf{k}) - \mathbf{u}_\mathsf{h}^\mathsf{k}|| + ||\mathbf{r}_\mathsf{h}(\mathbf{v}^\mathsf{k}) - \mathbf{v}_\mathsf{h}^\mathsf{k}||\right),$$

where first term is error between exact solution and projections, and second term is error between projections and approximate solution.

Let

$$\mathbf{e}_h^k(u) := \mathbf{r}_h(\mathbf{u}^k) - \mathbf{u}_h^k, \mathbf{e}_h^k(v) := \mathbf{r}_h(\mathbf{v}^k) - \mathbf{v}_h^k$$

and

$$e^k := ||\mathbf{e}_h^k(u)|| + ||\mathbf{e}_h^k(v)||.$$

We have

$$\mathsf{E}^\mathsf{k} \le \mathsf{C}_\mathsf{p}\mathsf{h}^2 + \mathsf{e}^\mathsf{k},$$

where

$$\mathsf{C}_\mathsf{p} := \mathsf{c}\left[\sup_\mathsf{t} ||\mathbf{u}(\mathsf{t})||_2 + \sup_\mathsf{t} ||\frac{\partial \mathbf{u}(\mathsf{t})}{\partial \mathsf{t}}||_2\right].$$

Convergence of approximation

Theorem 5. ***Convergence of Central difference approximation***
Let $(\mathbf{u}, \mathbf{v})$ *be the exact solution of peridynamics equation and Let* $(\mathbf{u}_h^k, \mathbf{v}_h^k)$ *be the FE approximate solution. If* $\mathbf{u}, \mathbf{v} \in C^2([0, T], H_0^2(D; \mathbb{R}^d))$*, then the scheme is consistent and the error* E^k *satisfies following bound*

$$\sup_{k \leq T/\Delta t} E^k = C_t \Delta t + C_s \frac{h^2}{\epsilon^2}$$

where constant C_t *and* C_s *are independent of* h *and* Δt *and depends on the norm of exact solution. Constant* L/ϵ^2 *is the Lipschitz constant of* $\mathbf{f}^\epsilon(u)$ *in* L^2*.*

Convergence of approximation

Outline of proof:

(1) Write peridynamics equation for projection $(\mathbf{r_h}(\mathbf{u^k}), \mathbf{r_h}(\mathbf{v^k}))$ which involves consistency error.

(2) Estimate consistency error terms. One of the error term is as follows

$$||\mathbf{f}^\epsilon(\mathbf{u}^k) - \mathbf{f}^\epsilon(\mathbf{r}_h(\mathbf{u}^k))|| \leq \frac{L}{\epsilon^2}||\mathbf{u}^k - \mathbf{r}_h(\mathbf{u}^k)||_{L^2} \leq \frac{Lc}{\epsilon^2}h^2 \sup_t ||\mathbf{u}(t)||_2$$

(3) Substract peridynamics equation corresponding to projection $(\mathbf{r_h}(\mathbf{u^k}), \mathbf{r_h}(\mathbf{v^k}))$ and approximate solution $(\mathbf{u_h^k}, \mathbf{v_h^k})$, use estimates on consistency errors, and apply discrete Grönwall inequality to obtain the bound on $\mathsf{e}^\mathsf{k} = ||\mathbf{u_h^k} - \mathbf{r_h}(\mathbf{u^k})|| + ||\mathbf{v_h^k} - \mathbf{r_h}(\mathbf{v^k})||$.

Stability of fully discrete approximation: Linearized peridynamic equation

Consider linearization of peridynamic force $\mathbf{f}^\epsilon$ defined as

$$\mathbf{f}_l^\epsilon(\mathbf{u})(\mathbf{x}) = \frac{4}{|B_\epsilon(\mathbf{x})|}\int_{B_\epsilon(\mathbf{x})} \omega(\mathbf{x})\omega(\mathbf{y})\frac{J^\epsilon(|\mathbf{y}-\mathbf{x}|)}{\epsilon} f'(0)S(\mathbf{u})\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}d\mathbf{y}.$$

Weak form of peridynamic equation is given by $(\rho\ddot{\mathbf{u}}(t), \tilde{\mathbf{u}}) + a_l^\epsilon(\mathbf{u}(t), \tilde{\mathbf{u}}) = (\mathbf{b}(t), \tilde{\mathbf{u}})$, where $a_l^\epsilon(\mathbf{u}, \mathbf{v})$ is now bilinear map.

Following **Karaa 2012**[1], wehave

Theorem 6. ***Stability of Central difference approximation of linearized peridynamics***
In the absence of body force $\mathbf{b}(t) = \mathbf{0}$ *for all* t, *if* Δt *satisfies the CFL like condition*

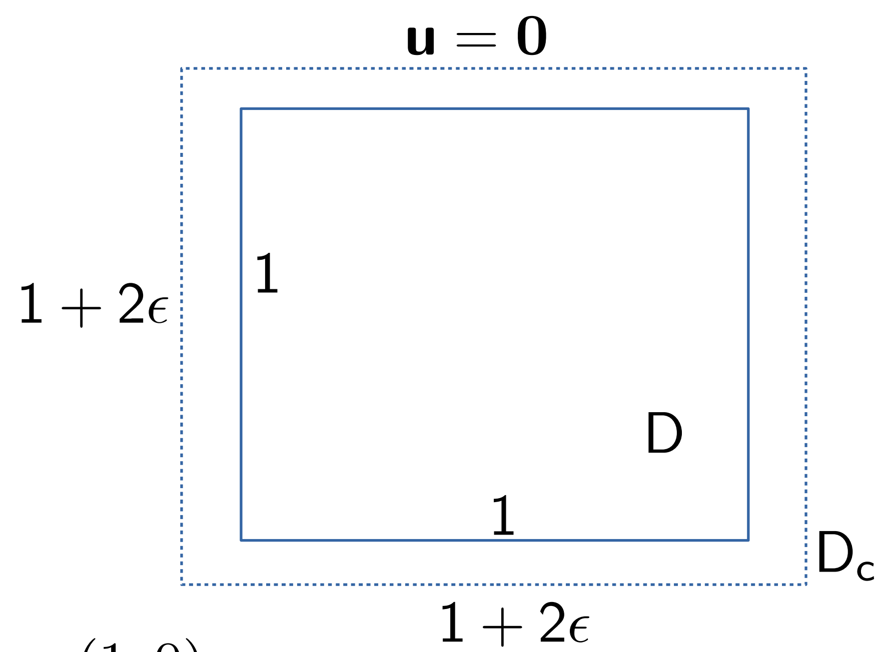
$$\frac{\Delta t^2}{4}\sup_{\mathbf{u}\in V_h\setminus\{0\}} \frac{a_l^\epsilon(\mathbf{u},\mathbf{u})}{(\mathbf{u},\mathbf{u})} \le 1,$$

then the discrete energy is conserved and we have the stability.

[1] Samir Karaa (2012) Stability and convergence of fully discrete finite element schemes for the acoustic wave equation. Journal of Applied Mathematics and Computing, 40(1-2):659–682, 2012.

Numerical results: Exact solutions[1]

- $\epsilon = 0.2$
- $h = \epsilon/4, \epsilon/8$
- Time domain $[0, 1]$ with $\Delta t = 2 \times 10^{-5}$
- $\rho = 1$, $f(r) = 1 - \exp[-r]$, $J(r) = 1 - r$



Let $\mathbf{w}(\mathbf{x}, t) = a(\mathbf{x}) \sin(n\pi(\mathbf{d} \cdot \mathbf{x} + t))\mathbf{d}$,

where $a(\mathbf{x}) = 0.001 * x_1 x_2 (1 - x_1)(1 - x_2), \mathbf{d} = (1, 0)$.

Define body force as follows

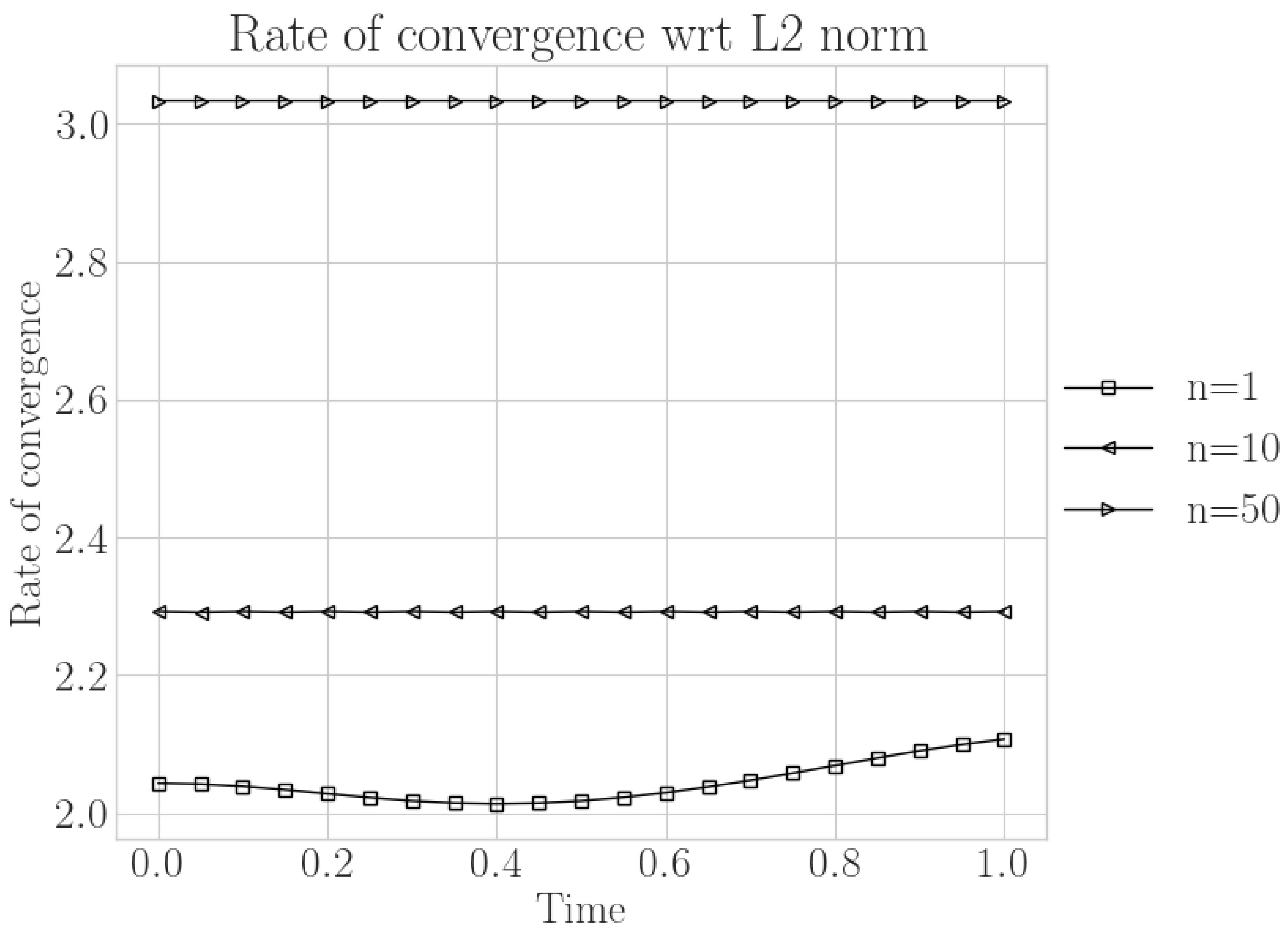
$$\mathbf{b}(\mathbf{x}) = \rho \partial^2_{tt} \mathbf{w}(\mathbf{x}, t) - \mathbf{f}^{\epsilon}(\mathbf{w}(t))(\mathbf{x})$$

and set initial condition $\mathbf{u}_0(x) = \mathbf{w}(0, \mathbf{x})$, $\mathbf{v}_0(x) = \dot{\mathbf{w}}(0, \mathbf{x})$.

Then $\mathbf{u}(\mathbf{x}, t) = \mathbf{w}(\mathbf{x}, t)$ is the solution.

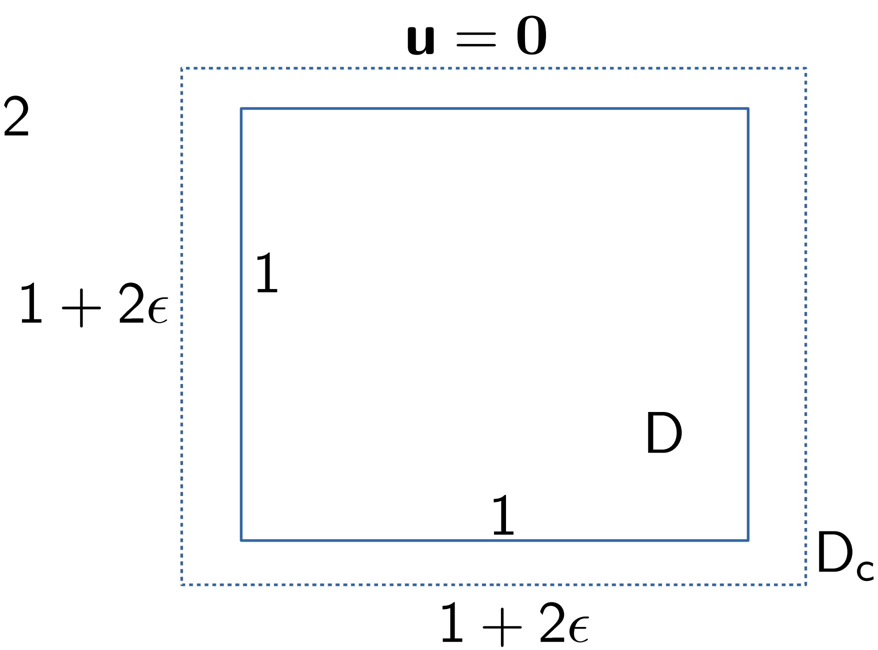
[1] Prashant K. Jha and Robert Lipton (2018) Robust finite element implementation for nonlocal fracture models. Under preparation.

Numerical results: Exact solutions



Numerical results: Different initial conditions

- $\epsilon = 0.2$
- $h = \epsilon/2, \epsilon/4, \epsilon/8$ with $r_h = h_1/h_2 = h_2/h_3 = 2$
- Time domain $[0, 1]$ with $\Delta t = 2 \times 10^{-5}$
- $\rho = 1$, $f(r) = 1 - \exp[-r]$, $J(r) = 1 - r$



$$\bar{\alpha} := \frac{\log(||\mathbf{u}_1 - \mathbf{u}_2||) - \log(||\mathbf{u}_2 - \mathbf{u}_3||)}{\log(r_h)},$$

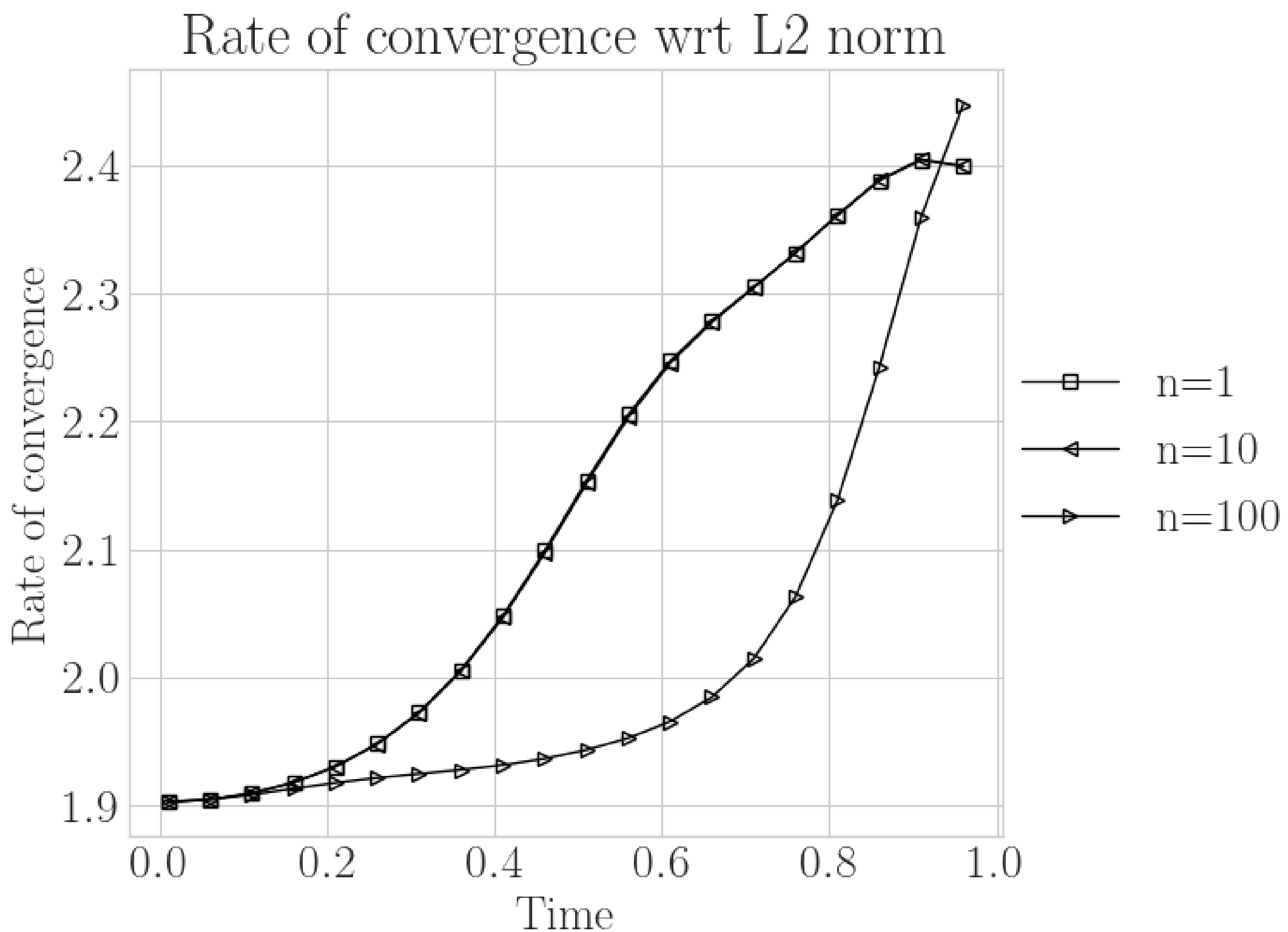
Let $\mathbf{u} = \mathbf{0}$ on D_c. We consider initial condition of the form

$$\mathbf{u}_0(\mathbf{x}) = \mathbf{0}, \ \mathbf{v}_0(\mathbf{x}) = n\pi a(\mathbf{x}) \exp[-|\mathbf{x} - \mathbf{x}_c|^2/b]\mathbf{d},$$

where $a(\mathbf{x}) = 0.1 * x_1 x_2 (1 - x_1)(1 - x_2)$ for $\mathbf{x} \in D$ and 0 otherwise,

$\mathbf{x}_c = (0.5, 0.5)$, $\mathbf{d} = (0, 1)$, and $b = 0.1$.

Numerical results: Different initial conditions

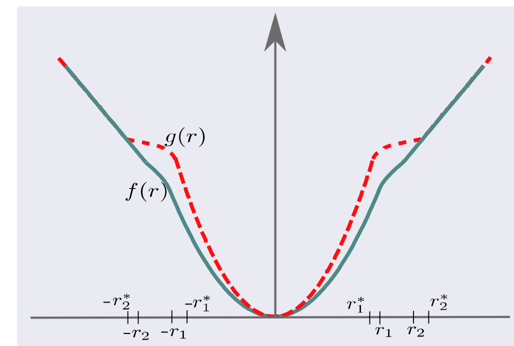


Numerical results for damage model

We introduce new damage model within peridynamic state-based framework in **Lipton et. al. 2018**[1].

We focus only on bond-based part of the interaction. Peridynamic force is of the form

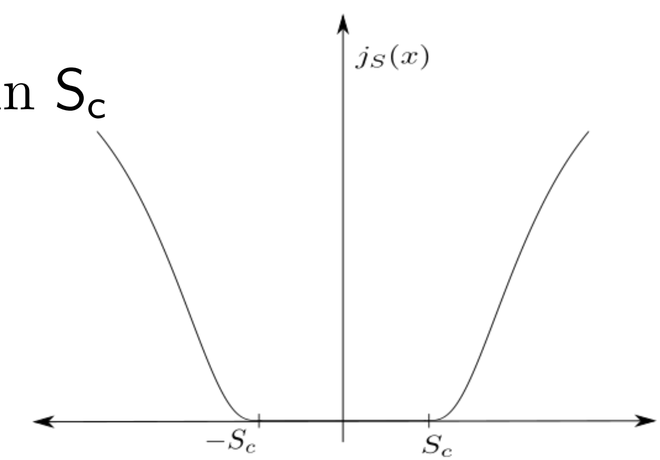
$$\mathbf{f}^\epsilon(\mathbf{x};\mathbf{u}(t)) = \frac{2}{|B_\epsilon(\mathbf{x})|}\int_{D\cap B_\epsilon(\mathbf{x})} H^T(\mathbf{u})(\mathbf{y},\mathbf{x},t)\hat{\mathbf{f}}^\epsilon(\mathbf{y},\mathbf{x};\mathbf{u}(t))\,d\mathbf{y},$$



where damage of bond $\mathbf{y}-\mathbf{x}$ at time t is given by

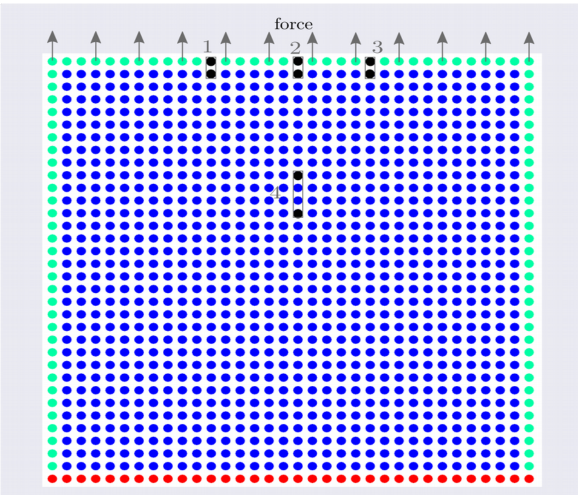
$$H^T(\mathbf{u})(\mathbf{y},\mathbf{x},t) = h\left(\underbrace{\int_0^t j_S(S(\mathbf{y},\mathbf{x},\tau;\mathbf{u}))\,d\tau}\right)$$

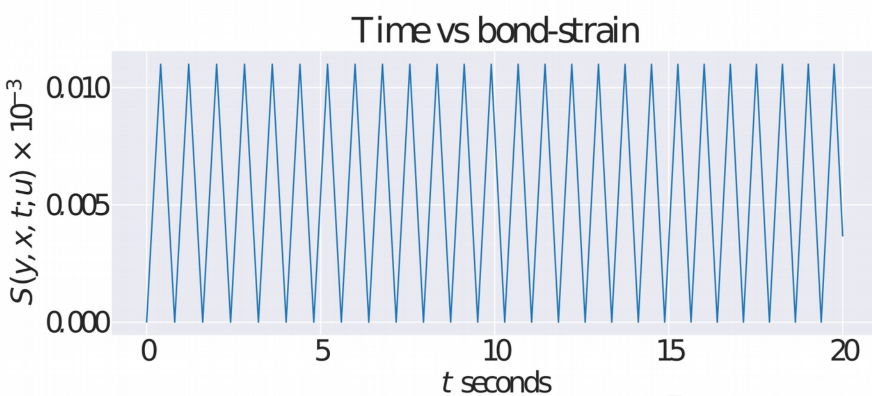
j_S is nonzero postive for strain only above critical strain S_c

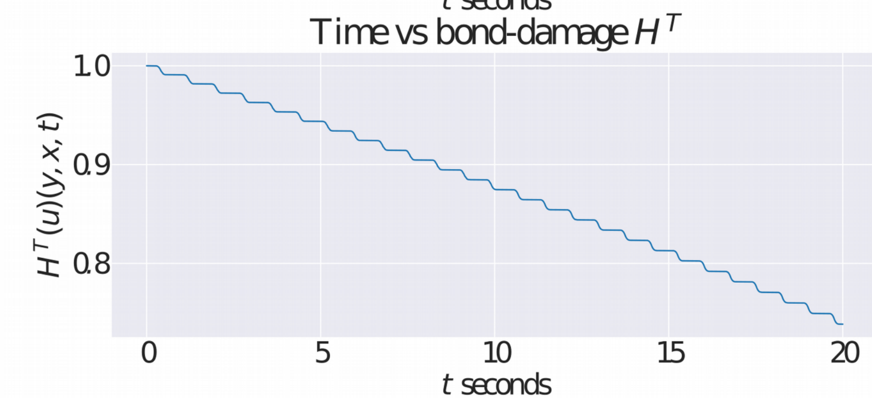


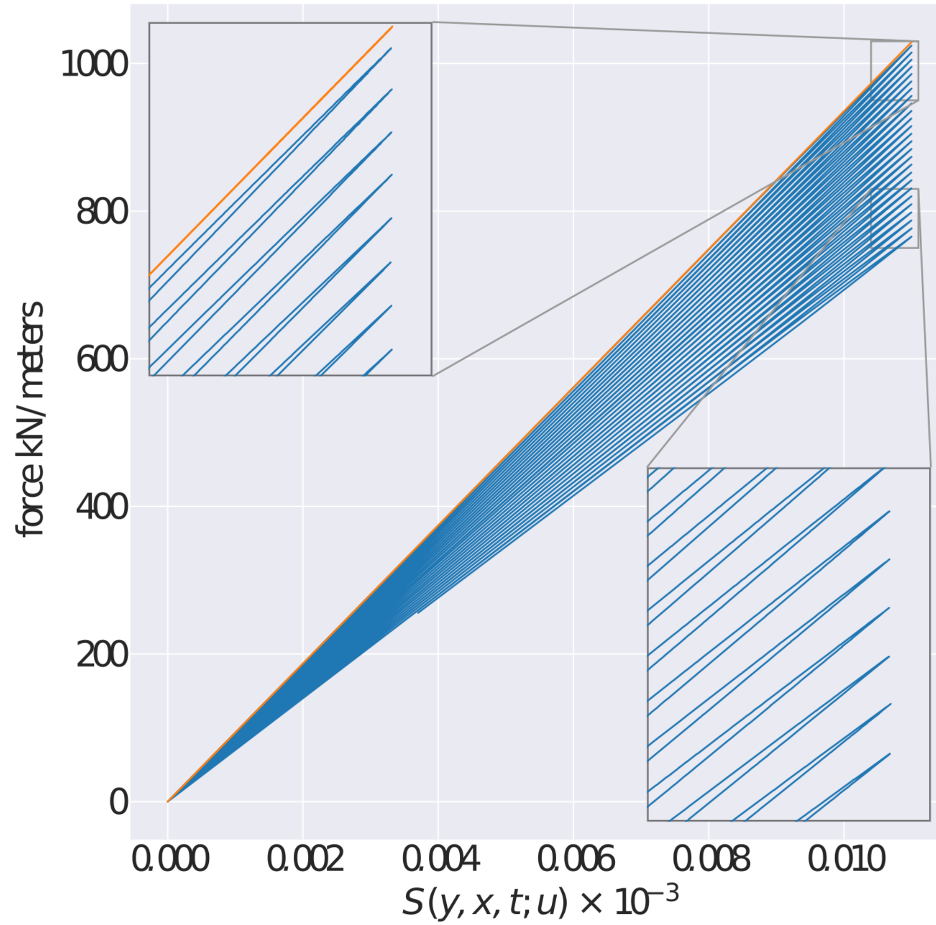
[1] Robert Lipton, Eyad Said and Prashant K. Jha (2018) Free Damage Propagation with Memory. Journal of Elasticity.

Periodic loading

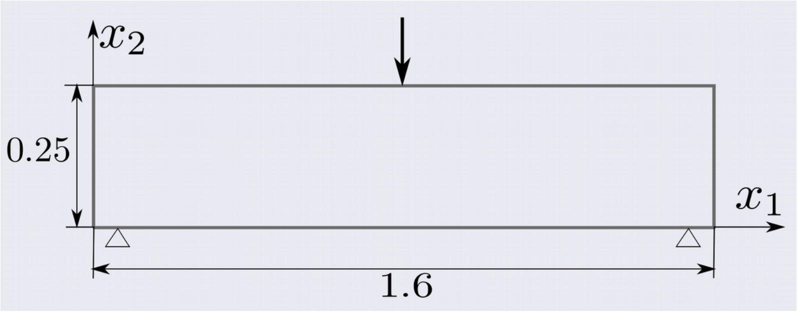


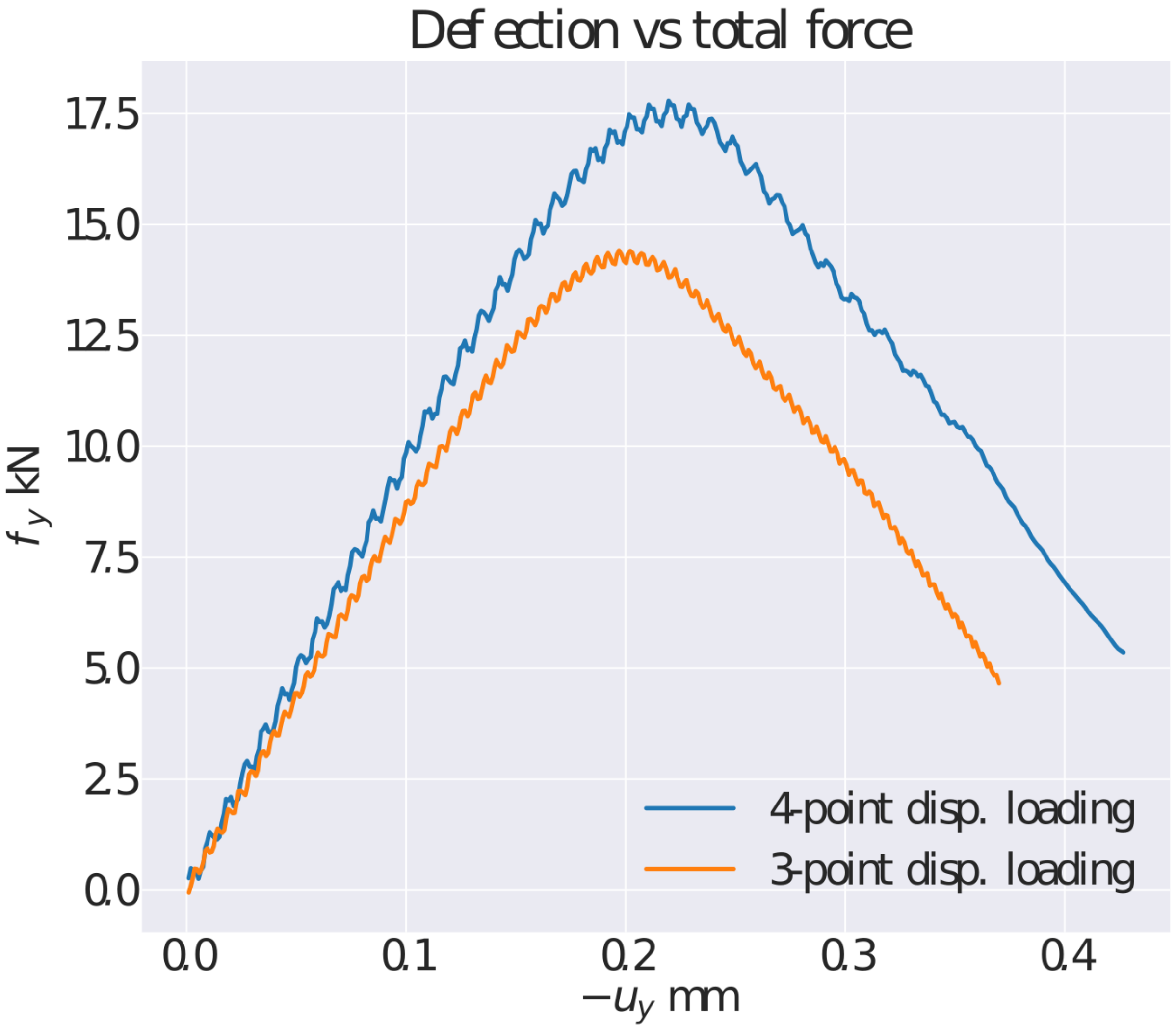






Bending test





Future works

- Numerical analysis of state-based model.
- Analysis of state-based peridynamic energy in the limit nonlocal length-scale tends to zero.
- Implementation of adaptive-mesh refinement.

Thank you!